GEODESY

INCLUDING

ASTRONOMICAL OBSERVATIONS, GRAVITY MEASUREMENTS, AND METHOD OF LEAST SQUARES

BY

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PREFACE TO THE FIRST EDITION

In this volume the author has attempted to produce a text-book on Geodesy adapted to a course of moderate length. The material has not been limited to what could be actually covered in the class, but much has been included for the purpose of giving the student a broader outlook and encouraging him to pursue the subject farther. Numerous references are given to the standard works.

Throughout the book the aim has been to make the underlying principles clear, and to emphasize the theory as well as the details of field work. The methods of observing and computing have been brought up to date so as to be consistent with the present practice of the Coast and Geodetic Survey.

The chapters on astronomy and least squares are included for the sake of completeness but do not pretend to be more than introductions to the standard works. The student cannot expect to master either of these subjects in a short course on geodesy, but must make a special study of each.

The author desires to acknowledge his indebtedness to those who have assisted in the preparation of this book, and especially to Professor J. W. Howard of the Massachusetts Institute of Technology for suggestions and criticism of the manuscript; to the Superintendent of the Coast and Geodetic Survey for valuable data and for the use of many photographs for illustrations; and to Messrs. C. L. Berger & Sons for the use of photographs of the pendulum apparatus and several electrotype plates. Tables XII to XVII are from electrotype plates from Breed and Hosmer's Principles and Practice of Surveying, Vol. II.

G. L. H.

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CONTENTS

CHAPTER I

GEODESY AND GEODETIC SURVEYING — TRIANGULATION

CHAPTER II

BASE-LINES

CHAPTER III

FIELD-WORK OF TRIANGULATION — MEASUREMENT OF HORIZONTAL ANGLES

CHAPTER IV

ASTRONOMICAL OBSERVATIONS

CHAPTER V

PROPERTIES OF THE SPHEROID

CHAPTER VI

CALCULATION OF TRIANGULATION

CHAPTER VII

CALCULATION OF GEODETIC POSITIONS

CHAPTER VIII

FIGURE OF THE EARTH

CHAPTER IX

GRAVITY MEASUREMENTS

GEODESY

CHAPTER I

GEODESY AND GEODETIC SURVEYING — TRIANGULATION

1. Geodesy.

Geodesy is the science which treats of investigations of the form and dimensions of the earth's surface. The methods which have been chiefly employed in determining the earth's figure are: — (1) the measurement of the lengths of *arcs* (meridians, parallels, or oblique arcs) on the surface of the earth, combined with the determination of the astronomical positions of points on these arcs; (2) the measurement of lengths in a network of triangles which covers an *area*, combined with the determination of astronomical positions; and (3) the measurement of the *variation* of the force of *gravity* in different parts of the earth's surface. Methods (1) and (2) give the form and size of the earth. Method (3) gives the form but not the absolute dimensions.

2. Geodetic Surveying.

Geodetic Surveying is that branch of the art of surveying which deals with such great areas that it becomes necessary to make systematic allowance for the effect of the earth's curvature. In making an accurate survey of a whole country, for example, the methods of plane surveying no longer suffice, and the theory of locating points and calculating their positions must be modified accordingly. Such surveys require the accurate location of widely separated points to control the accuracy of subsequent surveys for details, such as those for coast charts and topographic maps, or national and state boundaries. The general method

employed is that of triangulation, in which the location of points is made to depend upon the measurement of horizontal angles, the distances being calculated by trigonometry instead of being measured directly.

Although we may make this distinction when defining the terms it is not necessary to separate the two in practice, and in fact it is not usually possible to do so. It is evident that geodetic surveys must be made before accurate dimensions of the earth can be computed; and, conversely, it is true that before geodetic surveys can be calculated exactly the earth's dimensions must be known. Hence, geodetic surveys are usually conducted with a twofold purpose: (1) for collecting the scientific data of geodesy, and (2) for mapping large areas; every survey depending upon data previously determined, but also adding to or improving the data already existing. For this reason the measurements are often made with greater refinement than would be necessary for practical purposes alone.

The work of the geodesist is closely associated with various branches of other sciences, such as practical astronomy, geology, seismology, oceanography, and terrestrial physics. The activities of a geodetic survey usually include triangulation, astronomical observations, and leveling, for the control of surveys, and for studies of geodetic problems, topographic and hydrographic surveys, tidal and current observations, magnetic observations, seismological observations, gravity measurements, and cartography.

Some of the observations made by the geodesist reveal important facts about the internal constitution of the earth, and this information in turn is put to practical use in geological work. Methods used by the geodesist for studying variations in density of the earth's crust are also used for locating mineral wealth. The hydrographic charts are made to aid the navigator; horizontal and vertical control points and topographic maps are of assistance to engineers and others. Seismological studies are an aid to engineers and architects in designing structures to resist

earthquakes. Thus it is seen that geodesy is in close touch with science on the one hand and commerce and industry on the other.

3. Triangulation.

A triangulation system consists of a network of triangles the vertices of which are marked points (triangulation stations) on the earth's surface. It is essential that the length of one side of some triangle should be measured directly, and also that a sufficient number of horizontal angles should be measured to make possible the calculation of all the remaining triangle sides in the net. In addition to the measurements that are absolutely necessary for making these calculations it is important to have other measurements for the purpose of testing and verifying the accuracy of both the calculations and the field work. These may consist of check angles, additional base-lines, or observed astronomical azimuths.

It should be understood that after the field measurements of a triangulation are completed and computations made which reveal the errors in the angles and the distances, the entire system of triangulation is "adjusted" by the "method of least squares" so that all inconsistencies are removed from the final calculated results.

4. Classification of Triangulation.

There are two ways in which triangulation systems may be classified, and names are required for each. When classifying with reference to the use to which a system is put and its relation to other triangulation in the same system we speak of the larger triangles as the "main scheme" or the "principal triangulation," whereas the triangulation of lesser importance is called the "subsidiary" triangulation. When classifying according to the degree of accuracy finally secured we speak of the "order" of the triangulation. In 1925 the Board of Surveys and Maps of the Federal Government adopted for all branches of the government service the following classification:

	First order.	Second order.	Third order.	Fourth order.*
Triangulation	Average triangle closure 1″, check on base 1/25,000	Average triangle closure 3″, check on base 1/10,000	Average triangle closure 5″, check on base 1/5000	Graphic or transit angles

In this table are given the "errors of closure" of the triangle which may be used as a field test of the accuracy of the work. In first-order work the average error of closure of a triangle in the net should not be more than about 1″, and the maximum error of closure should not exceed 3″. The expression "check on base" is interpreted to mean that when a connection is made with a measured base, or with an adjusted line of a previously executed triangulation, and after the adjustment has been made for angle equations and side equations,† the remaining discrepancy between the computed and the measured lengths of this base shall not exceed one twenty-five thousandth part of the length of the base itself. The specifications for second-order and third-order work are to be interpreted similarly.

Formerly the words primary, secondary, and tertiary, were used to designate the different classes of triangulation. As these terms were used differently by different organizations and since they have been officially replaced it is not advisable to continue their use.

The function of the main scheme of triangulation in a survey is ordinarily to furnish the "control," that is, to furnish a sufficient number of accurately located points to make certain that the accuracy of the dependent triangulation and traverses does not fall below a certain specified amount. The subsidiary triangulation furnishes the points used immediately for the mapping work, or it may serve merely to connect these latter stations with stations in the main scheme.

* A committee of the American Society of Civil Engineers is now at work upon specifications for fourth order triangulation and traverse.

† See Art. 8 and Chapter XII.

The government program for horizontal control contemplates filling in belts of first-order triangulation about 100 miles apart, and second-order belts in between until there shall be no considerable area in the United States which is more than 25 miles distant from a triangulation station of the first or the second order.

4a. Traverses.

Under ordinary circumstances traverses are not so accurate as triangulation. In traversing, it is necessary to pass over the intervening ground, which may be extremely rough, and to measure distances under unfavorable conditions. Where the distance measurements are much inferior in accuracy to the angle measurements the traverse is necessarily less accurate than the triangulation. Formerly traverses were not regarded as sufficiently accurate for geodetic work. In recent years, however, traverses of the same, or nearly the same, order of accuracy as triangulation have been extensively used in flat, wooded country where triangulation would have been difficult and expensive.

The following specifications have been adopted by the Federal Board of Surveys and Maps for traverses used in government surveys:

First order. Measurements to correspond in accuracy with first-order triangulation; the check on the position of a point to be within one part in 25,000.* Directions to be kept accurate by azimuth observations made every 10 to 15 stations, the "probable error" to be about $0''.5$ of angle.

Second order. The distance check to be not less than 1/10,000; azimuths to be observed every 15 to 20 stations with a probable error of $1''.5$.

Third order. The distance check is to be 1/5000; the distances must be measured with standardized tapes, using spring balance and thermometer; azimuths to be observed every 30 to 50 stations with a probable error of $5''$.

* That is, the actual error in position must not be more than 1/25,000th part of the distance traversed, when the closure is made on an adjusted triangulation point.

Fourth-order traverse is for mapping purposes only and includes tape, stadia, or wheel traverse.

5. Length of Line.

The length of line which may be used in the main triangulation is determined largely by the character of the country to be surveyed. In California and Nevada, where the mountains are

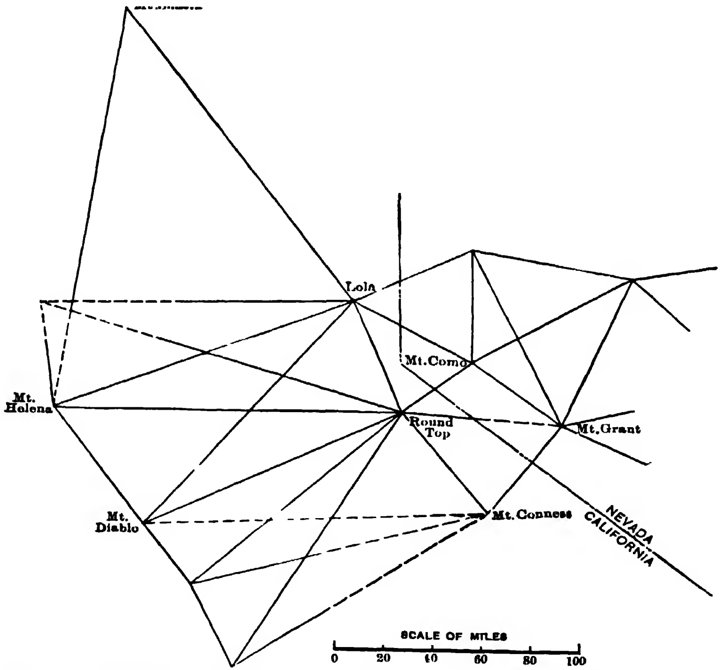

FIG. 1. Triangulation in California and Nevada (Davidson Quadrilaterals).

high and the atmosphere is exceptionally clear, the scheme of triangulation known as the "Davidson Quadrilaterals," (Fig. 1) contains lines from 50 miles to nearly 200 miles in length. The longest line sighted over was that joining Mt. Shasta and Mt.

Helena, a distance of 192 miles. The longest line used in the final computations was from Mt. Ellen to Uncompaghre Peak, a distance of 182.7 miles. In flat and wooded country the lines may be from 10 to 25 miles, and in some cases considerably shorter than this.

Although long lines appear to give more rapid progress, they are not necessarily the most economical or useful. So far as accuracy goes, it makes little difference whether lines are 5 kilometers or 300 kilometers in length. But points separated by such distances as the latter are not suitable for mapping purposes, and it would be necessary to locate a very large number of other stations before the detail work could begin. Another point to consider is that in an atmosphere which is not particularly clear the delays in observing over long lines may offset the advantages of a reduced number of stations. The decision should rest upon the following two considerations: 1. The cost of the work, including reconnoissance, signal building, and angle measuring should not be excessive, and 2. There should be a sufficient number of accessible stations established to serve the immediate purpose of the survey and also to leave points which will be useful to engineers. In other words, we should use that length of line which gives the maximum of usefulness combined with the minimum of cost.

6. Check Bases.

It has already been stated that at least one line in a system must be measured. In order to verify the accuracy of the triangulation it is customary to introduce additional base-lines at intervals varying from 50 to 500 miles. The lengths of these bases may be found by calculation of the triangles as well as by the direct measurement; this furnishes a most valuable check on the accuracy of the field work. In the triangulation of the United States Coast and Geodetic Survey the frequency with which bases should be measured is determined by the "strength factor" described on page 12. If the character of the country is such that a base may be measured almost anywhere then the factor

ΣR_1 between bases should be about 80. This will correspond to a chain of from 10 to 25 triangles, depending upon the strength of the figures secured. If conditions are such that it is difficult to measure a base in the desired location ΣR_1 may be allowed to approach but not to exceed 110. If, however, the factor is

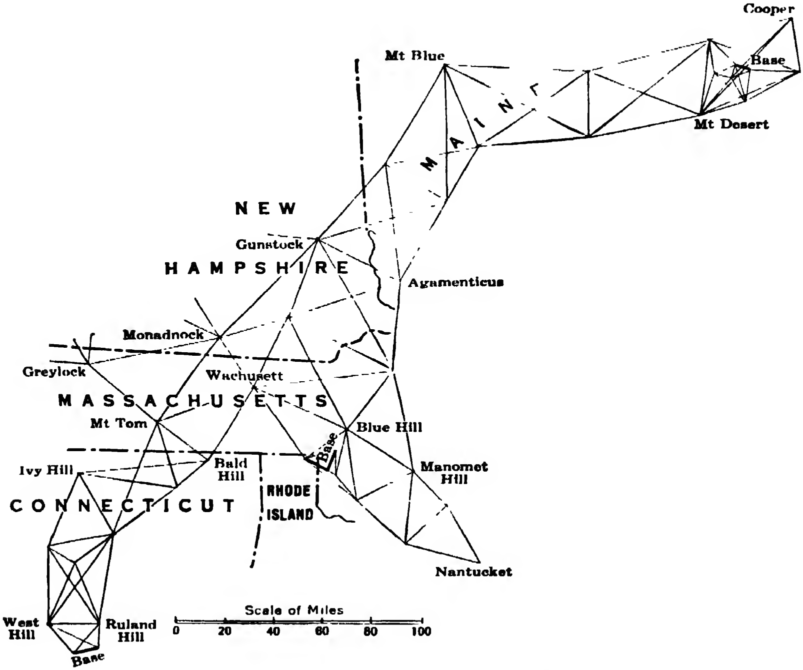

FIG. 2. Triangulation in New England.

close to this limit, there is danger that the check on the base will not meet the specified requirement and that in consequence another intermediate base will have to be measured.

In the triangulation of New England there are three bases: (1) the Fire Island base, about 9 miles long, measured in 1834; (2) the Massachusetts base, about 10 miles long, measured about

1844; and (3) the Epping base, about 5 miles long, measured in 1857. (Fig. 2.) The total length of the triangulation between the Epping and the Fire Island bases is about 400 miles. The accuracy with which the triangulation was executed is indicated by a comparison of the computed and measured lengths. The length of the Epping base as calculated from the Fire Island base is 0.042 meter less than the measured length; the length of the Epping base calculated from the Massachusetts base is 0.136 meter less than the measured length. This triangulation was executed long before the "strength factor" test came into use.

As another illustration of a belt of triangulation between two bases Fig. 2a shows that portion of the Texas-California arc which joins the Stanton base (Texas) with the Deming base (New Mexico).

7. Geometric Figures.

The geometric figure generally recognized as the best one for a belt of triangulation is the completed quadrilateral, consisting of four stations joined by six lines, thus forming four triangles in which there are altogether eight locally independent angles to be measured. This figure furnishes a large number of checks as compared with the number of angles measured, and gives a strong determination of length. The polygon having an interior station is also a strong figure; it is particularly well adapted to surveying an area rather than a belt. Figures which are more complex than these are apt to be expensive and troublesome to adjust, while single triangles do not furnish a sufficient number of checks and this results in diminished accuracy. In the work of the United States Coast and Geodetic Survey the principal triangulation is made up chiefly of quadrilaterals, and polygons having an interior station. In these figures all of the stations are supposed to be occupied with the triangulation instrument.

8. Strength of Figure.

In deciding which of several possible triangulation schemes to adopt it is important to inspect the different chains of geometric figures with a view to ascertaining which is the *strongest*, that is.

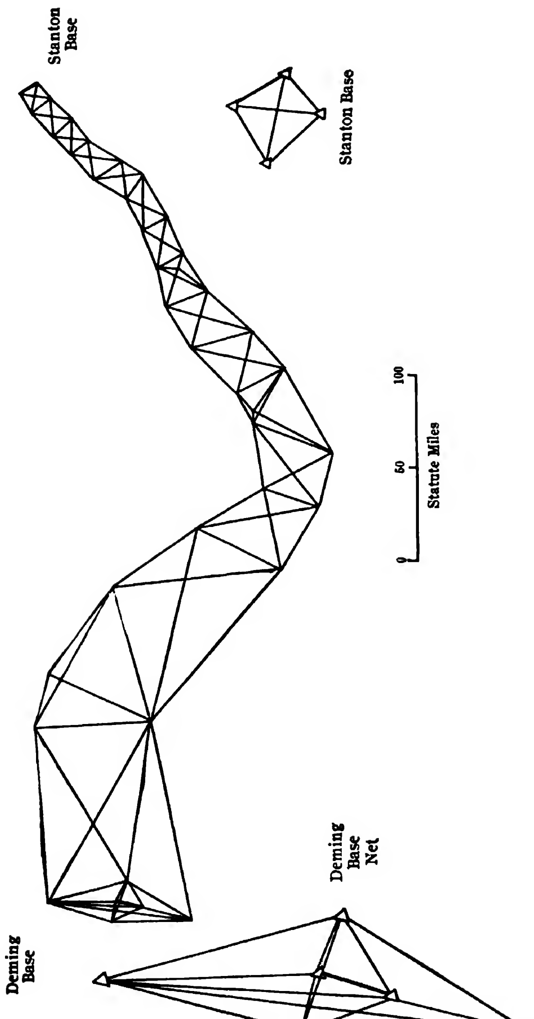

Fig. 2a. The Stanton Base and the Deming Base.

which one will give the calculated length of the final line with the least error due to the shape of the triangles and the composition of the figure.

An estimate of the uncertainty in the computed side of any triangle is given by its *probable error* as found by the method of least squares. If p is the probable error of the logarithm of the side AC of a triangle as calculated from the base BC, then

$$p^2 = \tfrac{2}{3}\, r^2(\delta_A{}^2 + \delta_A\delta_B + \delta_B{}^2).$$

in which δA and δB are the differences in the log sines of angles A and B corresponding to an angular difference of $1''$, and r is the average probable error of the adjusted *angles*. If d is the (average) probable error of an adjusted *direction*,* then

$$p^2 = \tfrac{4}{3}\, d^2(\delta_A{}^2 + \delta_A\delta_B + \delta_B{}^2).$$

A and B are the only angles used in the calculation of AC from BC and are called the *distance angles*. It is thus seen that the error of the computed side depends upon the size of the angles occurring in the triangle and that it may be computed from the differences for $1''$ as found in a table of log sines. The unit is ordinarily taken as one in the sixth decimal place.

Whenever the geometric figure is composed of several triangles connected in such a way that the length of the final line may be calculated by more than one route through the triangulation, the result obtained from such a figure is more accurate than that found by a simple chain of triangles. The greater the number of checks the greater the accuracy. In this case the square of the probable error of the log of any triangle side is given by the approximate formula

* When the horizontal angles of a triangulation are obtained by use of the so-called "direction theodolite" the angles are not determined separately as with the repeating instrument (or the ordinary transit), but the directions of signals are sighted and read in order around the horizon; the angles are obtained from the differences in direction. It is customary, therefore, for some purposes, to regard the directions of the different lines as observed independently, even though the measurements may have been made with a repeating theodolite. For the probable error of an observed direction see table on p. 97.

$$p^2 = \tfrac{4}{3}\, d^2 \cdot \frac{D - C}{D} \cdot \sum [\delta_A{}^2 + \delta_A \delta_B + \delta_B{}^2]$$

in which

D = the number of new directions observed

C = the number of geometric conditions that must be satisfied in the figure

δ_A and δ_B = the differences for 1″ in the log sines of the angles A and B in units of 6th decimal place

d = the probable error of an observed direction*

Σ = the *sum* of all the quantities in brackets.

In each separate triangle, δ_A and δ_B must be taken out for the two distance angles, whatever their designation may be.

Evidently the factor $\frac{D - C}{D}$ depends upon the nature of the figure chosen, and the quantity in brackets depends upon the shape of the triangle in question. The product of the two (called the *strength factor*, R) is a measure of the strength of the figure and is independent of the precision with which the angles themselves are measured. The strength, R, of any figure is given by

$$R = \frac{D - C}{D} \cdot \sum [\delta_A{}^2 + \delta_A \delta_B + \delta_B{}^2].$$

The smaller the value of R the stronger the figure, that is, the smaller the error in the computed length of the final line.

If the value of this factor be computed for every possible route through any one triangulation system there will result a minimum value (R_1) for the best chain, a second best value (R_2), a third and a fourth and so on. It will be found that the chain of triangles having the greatest influence in fixing the final length of the line (after adjustment) is that corresponding to R_1, or the best chain. The second best (R_2) has some influence, and the others correspondingly less. In other words, when the system is adjusted the angles of the R_1 chain will receive smaller cor-

* See page 97.

rections than those of the R_2 chain, and so on. Hence in choosing between two or more possible systems of triangulation which join a given line with a given base, that route having the smallest R_1 is to be preferred (so far as accuracy is concerned) unless R_1 proves to be nearly the same for the different routes, in which case that chain having the smallest R_2 would be chosen.

As an example of the way in which the preceding method would be applied, take the case of the quadrilateral shown in Fig. 3. Assuming the base AB to be a line *already fixed in direction and length*, the point C is then determined by observing the *new* directions AC and BC. D is fixed by the directions AD and

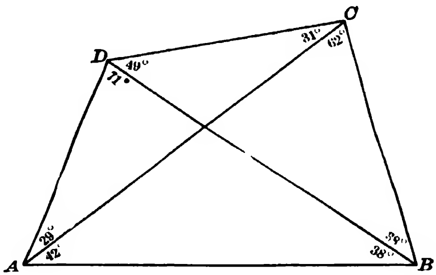

FIG. 3.

BD. In addition to these four, the directions CB, CA, CD, DC, DB, DA, are all observed. This gives 10 observed (new) directions as the value of D in the formula. In determining the number of geometric conditions, C, it is seen that there are four triangles, and that in each one the sum of the three angles must equal a fixed amount, 180° + the spherical excess of the triangle. It will be found, however, that if any three of these triangles are made to fulfill these conditions, the fourth will necessarily do so, and hence is not really independent; in other words there are three conditions dependent upon the closure of the triangles. In addition to these three angle conditions there is also a distance check, that is, the angles must be so related that the computed distance CD will come out the same, no matter which triangles

are used in making the computation. The angles in each triangle may in each case add up to the correct amount, and yet the figure will not be a perfect quadrilateral unless this last condition is also fulfilled. There are then, in all, four geometric* conditions existing among the angles. ($C = 4$.) The factor for the completed quadrilateral is, therefore,

$$\frac{D - C}{D} = \frac{10 - 4}{10} = 0.60.$$

In the triangle ACB (Fig. 3) the distance angles for computing AC are $A = 62°$ and $B = 76°$. The difference for $1''$ for $62° = 1.12$ and for $76° = 0.53$, in units of the sixth decimal place. The quantity in brackets in the formula is, therefore,

$$((1.12)^2 + 1.12 \times 0.53 + (0.53)^2) = (1.25 + 0.59 + 0.28) = 2.12,$$
or 2 to the nearest unit.

In Table I these numbers are given for all combinations of angles which will occur in practice, so that this factor may be found at once by entering the table with the two distance angles. Ordinarily the smaller angle will be found at the top and the larger angle at the side. Accurate interpolation is unnecessary. For the triangle DCA the angles are 29° and 120°, the corresponding number being 11. Therefore, the value of R is $0.60 \times (2+11) = 7.8$. For the triangle ADB the angles are 71° and 71°, and the number is 2. For triangle BDC the angles are 38° and 93°, the number being 7. The corresponding value of R is $0.60 \times (2 + 7) = 5.4$. For triangles ACB and DCB, $R = 15.6$; for triangles DBA and DCA, $R = 30.6$. Therefore, we have

$$\begin{aligned} R_1 &= 5.4 \\ R_2 &= 7.8 \\ R_3 &= 15.6 \\ R_4 &= 30.6 \end{aligned}$$

* The above mentioned condition equations are not the only ones that might be selected and used for adjusting a quadrilateral. The total number, however, is always four.

In comparing the strength of this quadrilateral with that of any other figure reliance would be placed mainly upon $R_1 = 5.4$ and partly upon $R_2 = 7.8$.

Following are the values of $\frac{D - C}{D}$ for several figures frequently occurring in triangulation: — single triangle, 0.75; completed quadrilateral, 0.60; quadrilateral with one station on fixed line not occupied, 0.75; quadrilateral with one station not on fixed line not occupied, 0.71; triangle with interior station, 0.60; triangle with interior station, one station on fixed line not occupied, 0.75; triangle with interior station, one station not on fixed line not occupied, 0.71; four sided figure with interior station, 0.64; five sided figure with interior station, 0.67; six sided figure with interior station, 0.68. For additional cases see U. S. Coast and Geodetic Survey special publications 26 and 120.

From the preceding discussion it will be seen that if a small angle occurs opposite the base or the side being computed the determination is weak, as indicated by a large R factor. If this is R_1, the figure is weak. But if the small angle is not opposite either the base or the advance side, the figure may be strong. Also, if the small angle occurs in R_3 or R_4 the figure may be strong.

It should be remembered that the above criterion is a test for *length only,* and not for azimuth or absolute position.

Another point to be noticed is that a figure may be strong and yet advance so slowly in the general direction of the triangulation as to make it an uneconomical one to use.

Figure 4 illustrates a figure that is strong, but gives slow progress in the general direction of the triangulation, while Fig. 5 is weak but advances rapidly. Figure 6, although badly distorted is fairly strong. Figure 7 is strong. Figure 8 and Fig. 9 are examples of good expansions from bases. In Fig. 8 the ratio of expansion is 2 to 1. The R_1 factor 5 indicates that this is a strong figure. Figure 9 in which $R_1 = 9$ is also a good figure for expanding a base.

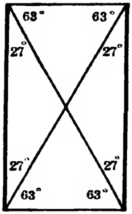

FIG. 4.

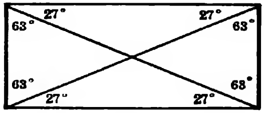

FIG. 5.

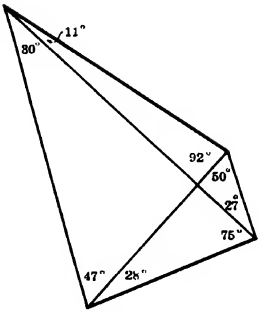

FIG. 6.

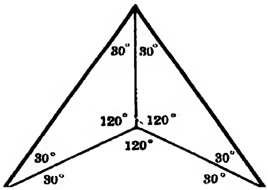

FIG. 7.

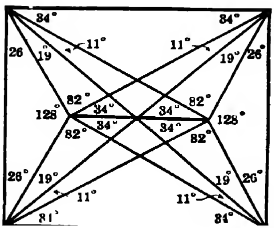

FIG. 8.

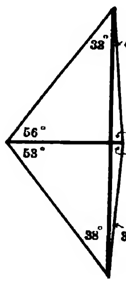

FIG. 9.

9. Number of Conditions in a Figure.

In determining the number of conditions in any figure it is well to proceed by plotting the figure, point by point, and writing down the conditions as they occur, but it will be of assistance to have a check on the results obtained by this process. If n represents the total number of (locally independent) angles measured, and s the number of stations, then, since it requires two angles to locate a third point from the base-line, two more to locate a fourth point from any two of these three points, and so on, the number of angles required is $2(s-2)$; and since each additional angle gives rise to a condition, the number of conditions will equal the number of *superfluous angles*, or

$$\begin{aligned} C &= n - 2(s-2) \\ &= n - 2s + 4. \end{aligned}$$

For example, in a quadrilateral in which one station is unoccupied there are six angles measured, and

$$C = 6 - 8 + 4 = 2.$$

The number of conditions may also be found from the equation

$$C = 2l - l_1 - 3s + s_u + 4$$

where l = the total number of lines
l_1 = the number of lines sighted in one direction only
s = the total number of stations
s_u = the total number of unoccupied stations.

In the preceding example this equation becomes

$$C = 12 - 3 - 12 + 1 + 4 = 2.$$

10. Allowable Limits of R_1 and R_2.

In triangulation of the first order the value of R for any one figure (between base nets) should not exceed 25 in the best chain (R_1) and 80 in the second best chain (R_2); and it is desirable to keep the values far below these limits.

In general a new base should be measured whenever R_1 amounts to 80. This will correspond to a chain of from 10 to 25 triangles,

according to the strength of the figures chosen. If it is difficult to secure a base site at the point indicated ($\Sigma R_1 = 80$) then ΣR_1 may be as great as 110 but must not exceed this amount. For first order accuracy the base check must not exceed one part in 25,000, so if ΣR_1 is allowed to approach the higher limit there is danger that a new intermediate base may become necessary.

11. Reconnoissance

The work of planning the system is in many respects the most important part of the project and demands much experience and skill. Upon the proper selection of stations will depend very largely the accuracy of the result, as well as the cost of the work. No amount of care in the subsequent field-work will fully compensate for the adoption of an inferior scheme of triangulation. Three points in particular will have to be kept in mind in planning a survey: (1) the "strength" of the figures adopted; (2) the distribution of the points with reference to the requirements of the subsequent detail surveys; and (3) the cost of the work. In deciding which stations to adopt it is desirable to make a preliminary examination of all available data, such as maps and known elevations. If no map of the region exists, a sketch map must be made as the reconnoissance proceeds. While much information may be obtained from such maps as are available, the final decision regarding the adoption of points must rest upon an examination made in the field. All lines should be tested to see if the two stations are intervisible. This may be done by means of field glasses and heliotrope signals. In cases where the points are not intervisible, owing to intervening hills or to the curvature of the earth's surface, it will be necessary to determine approximately, by means of vertical angles or by the barometer, the elevation of the proposed stations and of as many intermediate points as may be required, and then to calculate the height to which towers will have to be built in order to render the proposed stations visible. If the height of the towers is such as to make the cost prohibitive, the line must be abandoned and another scheme of triangles substituted.

12. Calculation of Height of Observing Tower.

After determining the elevations of the stations and the intervening hills along a line, as well as the distances between them, the height of the tower required may be found by the following method: The curvature of the earth's surface causes all points to appear lower than they actually are. A hill appearing to be exactly on the level of the observer's eye is in reality higher above sea-level than the observer. The light coming from the hill to the observer's eye does not, however, travel in a straight line, but is bent, or refracted, by the atmosphere into a curve which is concave downward and is approximately circular. The result is that the object appears higher than it would if there were no refraction. The amount of the apparent change in height due to refraction is found to be only about one-seventh part of the apparent depression due to curvature. Since these two corrections always have opposite signs and have a nearly fixed relation to each other, it is sufficient in practice to calculate the correction to the difference in height due to both curvature and refraction, and to treat the combined correction as though it were due to curvature alone, since the curvature correction, being the larger, always determines which way the total correction shall be applied.

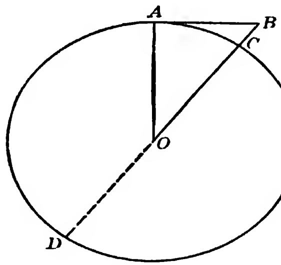

FIG. 10.

In Fig. 10, A is the position of the observer, looking in a horizontal direction toward point B. BC is the amount by which B appears lower than it really is, since A and C are both at the same elevation (sea-level).

By geometry, $BC : AB = AB : BD$

or
$$BC = \frac{AB^2}{BD}.$$

Since BC is small compared with BD, the percentage error is

small if we call $AB = AC$ and $BD =$ the diameter of the earth, whence

$$BC = \frac{(\text{dist.})^2}{\text{diameter}} \text{ (approx.).}$$

The light from B' (Fig. 11) follows the dotted curved path which is tangent to the sight line at A. The observer therefore sees B' at B. In order to find the relation of BB' to BC it is convenient to employ m, the *coefficient of refraction*, which is defined as the number by which the central angle AOB must be multiplied in order to obtain the angle BAB'; therefore

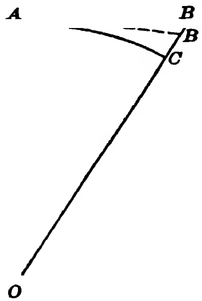

FIG. 11.

$$\begin{aligned}\text{angle of refraction} &= BAB' \\ &= m \times AOB \\ &= m \times 2\,BAC.\end{aligned}$$

Since these angles are small, distances BB' and BC are nearly proportional to the angles themselves, hence

$$BB' : BC = BAB' : BAC$$

and

$$BB' = 2\,m \times BC.$$

The net correction ($B'C = h$) is the difference between the two, that is

$$\begin{aligned}h &= BC - BB' \\ &= \frac{(\text{dist.})^2}{\text{diam.}} - 2\,m\,\frac{(\text{dist.})^2}{\text{diam.}} \\ &= \frac{(\text{dist.})^2}{\text{diam.}}(1 - 2\,m).\end{aligned}$$

The mean value of m is found to be about 0.070. Substituting this, and the value for the earth's diameter, and reducing h to feet, we have

$$h \text{ (in feet)} = K^2 \text{ (in miles)} \times 0.574,$$

or

$$K \text{ (in miles)} = \sqrt{h \text{ (in ft.)}} \times 1.32,$$

in which K is the distance in miles. Values of h and K for distances up to 60 miles will be found in Table II.

As an example of how this formula is applied, suppose it is desired to sight from A to B (Fig. 12), and that a hill C obstructs the line. At A draw a horizontal line AD and also a curve AE

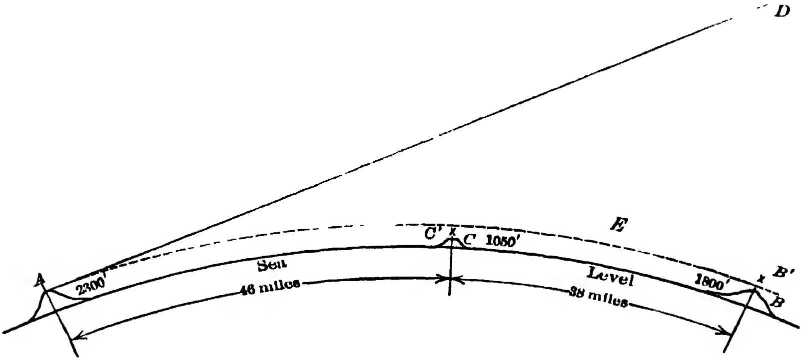

FIG. 12

parallel to sea-level. The distance from the tangent to the dotted curve at C is $\frac{K^2}{\text{diam.}}$, which for 46 miles is 1411.9 ft. Similarly, at B, $\frac{K^2}{\text{diam.}} = 4708.0$ ft. But since the ray of light from B to A is curved, B is seen at B', or 659.2 ft. nearer to the tangent AD; similarly, C appears to be 197.7 ft. nearer the tangent line. Therefore, in deciding the question of visibility we may compute the combined correction and say at once that the curve at C is 1214.2 ft. below AD, and at B is 4048.8 ft.* below AD. Adding 2300 ft. (the elevation of A) to each of these values of h, we obtain the (vertical) distances from the tangent line down to sea-level, namely 3514.2 ft. and 6348.8 ft. at C and B, respectively. Subtracting the elevations of C and B, we obtain 2464.2 ft. and

* Since the table extends only to 60 miles, the value of h is first found for half the distance (42 mi.), and the result multiplied by 4.

4548.8 ft. as the distances of points C and B below the tangent line AD. The three points are now referred to a straight line (the tangent), and the question of visibility is determined at once by similar triangles. In Fig. 13 it will be seen that the straight line from B' to A is $\frac{46}{84} \times 4548.8 = 2491.0$ ft. below the

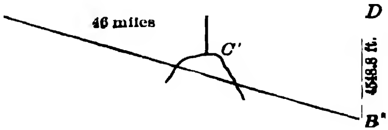

FIG. 13.

tangent (opposite C), and consequently is 26.8 ft. lower than C. Twenty-seven-foot towers would therefore barely make B' visible from A. In order to avoid the atmospheric disturbances near the ground at C the towers would really have to be carried up to a height of 40 ft. or even more. Of course the line of sight is not actually straight between A and B, as shown in the diagram; but this method of solving the problem gives the same result as though the curvature and refraction were dealt with separately and the sight lines all drawn curved. Notice that the two corrections are never separated in practice, but are used as combined in the table like a single correction.

If it were required to find the heights of towers necessary to make it possible to sight from A across a water surface to D, we

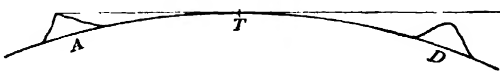

FIG. 14.

should proceed as follows: Suppose the elevation of A above the water surface is 20 ft. and that of D is 10 ft. From A we may draw a line tangent to the water-level at T (Fig. 14). Knowing the height of A, we may find the distance AT from Table II. Subtracting this distance from AD, we find the distance TD.

From this latter distance we may compute the height of the tangent line above the surface at D, and, finally, knowing the height of D, we find the distance of D below the tangent line. Now that the points are referred to a straight line, we have at once the height of tower required on D alone. If the two towers are to be of equal height, we may estimate the required height closely and then verify the result by a second computation, adding the assumed height of the tower to the elevation of A.

If it is desired to keep the line of sight at least 10 ft. above the surface at every point in order to avoid errors due to excessive refraction, we may draw a parallel curve 10 ft. above the water surface and solve the problem as before. The difference in radii of the two curves will not have an appreciable effect on the computed values of h and K.

13. Method of Marking Stations.

The importance of permanently marking a trigonometric station and connecting it with other reference marks cannot be easily overestimated, since by this means we may avoid the costly work of reproducing triangulation points which have been lost.

When the station is on ledge, the point is best marked by making a fairly deep drill hole and setting a copper bolt into it. A triangle is chiseled around the hole as an aid in identifying the point. Other drill and chisel marks should be made in the vicinity, and their distances and directions from the center mark determined; these will serve as an aid in recovering the position of the center mark in case it is lost.

If the station is on gravel or other soft material, the station mark on the surface is usually a stone or concrete post, set deep enough to be unaffected by frost action and having a drill hole or other distinguishing mark on top. There is usually also a subsurface mark, such as a second stone post, a bottle or a circular piece of earthenware, placed some distance below the surface mark, to preserve the location in case the latter is lost. The Coast and Geodetic Survey and the United States Geological Survey use cast bronze discs provided with a shaft ready to place

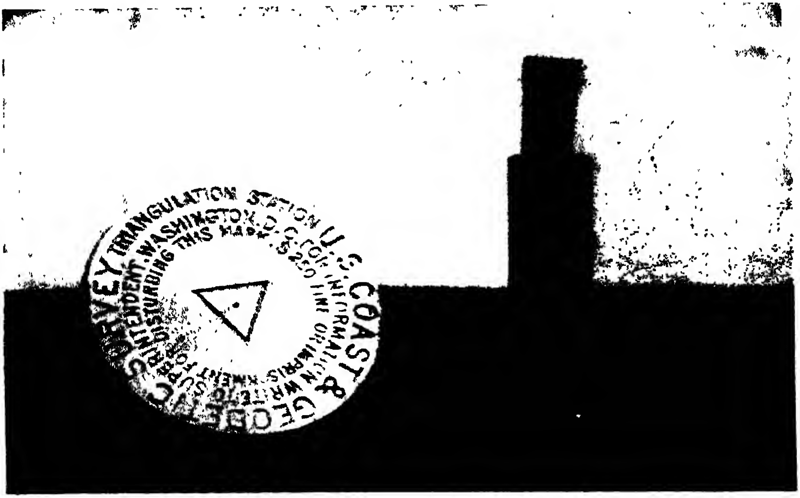

FIG. 15. Triangulation Station Mark. (*Coast and Geodetic Survey.*)

FIG. 16. Reference Mark. (*Coast and Geodetic Survey.*)

in concrete, and bearing an inscription giving the name of the organization and other information. (See Figs. 15 and 16.)

The following description and sketch are given to illustrate a description of a triangulation station.

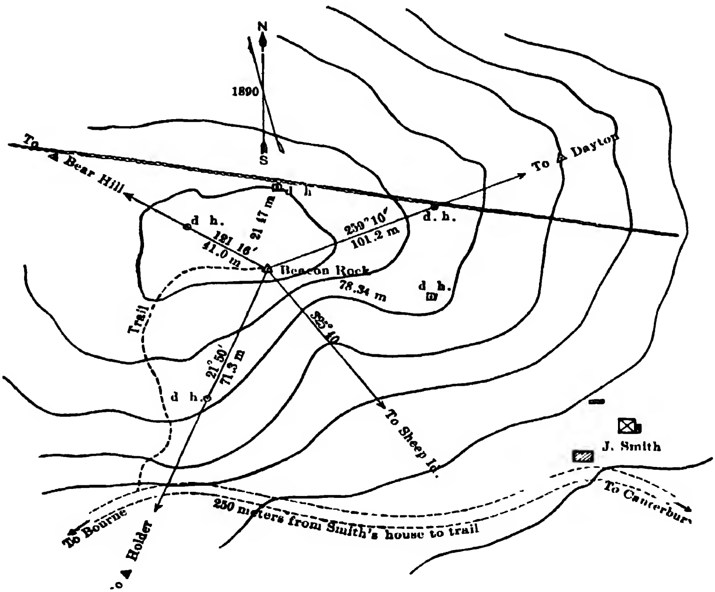

FIG. 17. Sketch of Triangulation Station.

Triangulation Station "Beacon Rock."

The station is in the town of ________, ________, on a hill on the property of John Smith situated on the north side of the road from Bourne to Canterbury. It is reached by a trail which leaves the road at a point about 250 meters west of Smith's house. It is about 225 meters by trail to the station. The point is marked by a one-inch copper bolt set in a drill hole in the ledge and with a triangle chiseled around it, and by witness marks as shown in the accompanying sketch and table. The hill is somewhat wooded to the north and west, but there is a clear view in all other directions.

DISTANCES AND AZIMUTHS FROM CENTER

Station.	Azimuth.	Dist. to drill hole.
Holder	21° 50′	71.3 m.
Bear Hill	121° 16′	41.0 m.
Witness Mark	185° 30′	21.47 m.
Dayton	259° 10′	101.2 m.
Witness Mark	283° 05′	78.34 m.
Sheep Id.	325° 40′	....

14. Signals, Tripods.

In order that the exact position of the station may be visible to the observer when measuring the angles, a signal of some sort is erected over the station. For comparatively short lines, less

FIG. 18. Tripod Signal.

than about 10 miles, the tripod signal is often sufficient. (See Fig. 18.) It is not expensive to build, saves the cost of a man to attend signal lights (as is necessary with heliotropes or electric lights), and permits setting the instrument over the station with-

out removing the signal. It usually consists of a mast of $4'' \times 4''$ spruce, with legs of about the same size. Three horizontal braces of smaller dimensions ($2'' \times 3''$) tie the mast to the legs, and three longer horizontal braces are nailed to the legs. If the signal is very large, additional sets of braces may be put on, to give greater stiffness. The size of the mast may be increased by nailing on one-inch boards, giving a mast $6'' \times 6''$.

The objection to such a signal for precise work is that there is almost always some error due to "phase," that is, stronger illumination on one side than on the other. It is nearly always possible for the observer to see two sides of the squared pole, and one side is usually better illuminated than the other, so that the pointings with the theodolite are not directed toward the center of the pole, but toward the center of the part that is distinctly visible. This error may be avoided, if it is not necessary to point at the signal from more than one station at one time, by using for a target a board facing exactly toward the observer and centered exactly over the station mark. Even the small error due to thickness of the board may be avoided by bevelling the edges. This must be shifted each time it is sighted on from a new station.

15. Heliotropes.

When sighting over longer lines it is necessary to use heliotrope signals if observing by day, and electric lights if observing by night. The heliotrope is simply a plane mirror with some device for pointing it so that reflected sunlight will reach the distant station. The two more common heliotropes are (1) the one in which the light is directed through two circular rings of slightly different diameters (Fig. 19), and (2) that known as the Steinheil heliotrope (Fig. 20).

The ring heliotrope consists essentially of two circular metal rings, of slightly different diameters, mounted on a frame, and a mirror mounted in line with the two rings in such a manner that it can be moved about two axes at right angles to each other. For convenience in observing distant stations these two rings and

the mirror are often mounted on the barrel of a telescope. The rings should be so mounted that the line between the centers of the rings may be adjusted parallel to the line of sight of the telescope. In using the heliotrope the axis of the rings is pointed by means of threads which mark the center of the openings, or by means of the telescope itself after the axis of rings and the line of sight of the telescope have been made parallel. Since the sun's

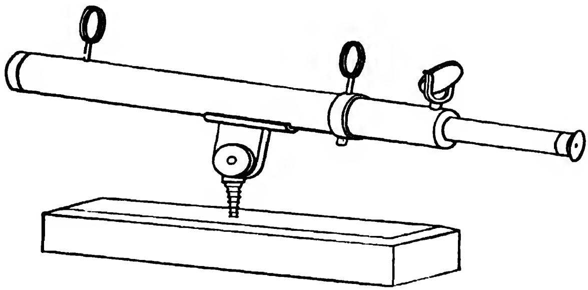

Fig. 19. Ring Heliotrope

apparent diameter is about $0° 32'$, the angle of the cone of rays reflected from the mirror is also $0° 32'$. It is not necessary, therefore, to point the beam of light with great precision. If the central ray is nearly a quarter of a degree to one side of the station, there will still be some light visible to the observer at the distant station. On account of the rapidity of the sun's motion it is necessary to reset the heliotrope mirror at intervals of one minute or less.

The Steinheil heliotrope consists of a mirror with both faces ground plane and exactly parallel and so mounted that it can be moved about two axes at right angles to each other. One of these axes is coincident with that of a cylindrical tube which contains a small biconvex lens and a white surface (usually plaster of Paris) for reflecting light. This tube may be moved about two other axes at right angles to each other. A small circular portion of the glass in the center of the mirror is left

FIG. 20. Steinheil Heliotrope.

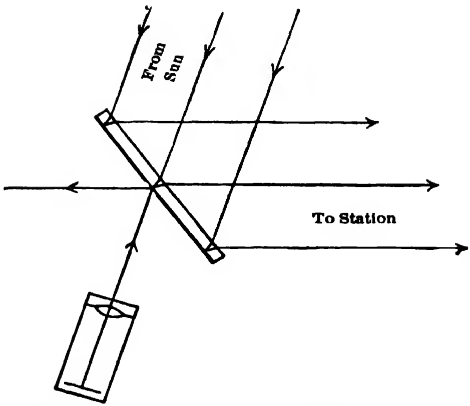

FIG. 21. Showing Optical Principle of the Steinheil Heliotrope.

unsilvered, so that light may pass through the glass plate down into the tube.

In pointing the Steinheil heliotrope the cylindrical tube containing the lens must be pointed toward the sun, so that the light which passes through the hole in the mirror will pass through the lens, and, after reflection from the plaster surface, will again pass

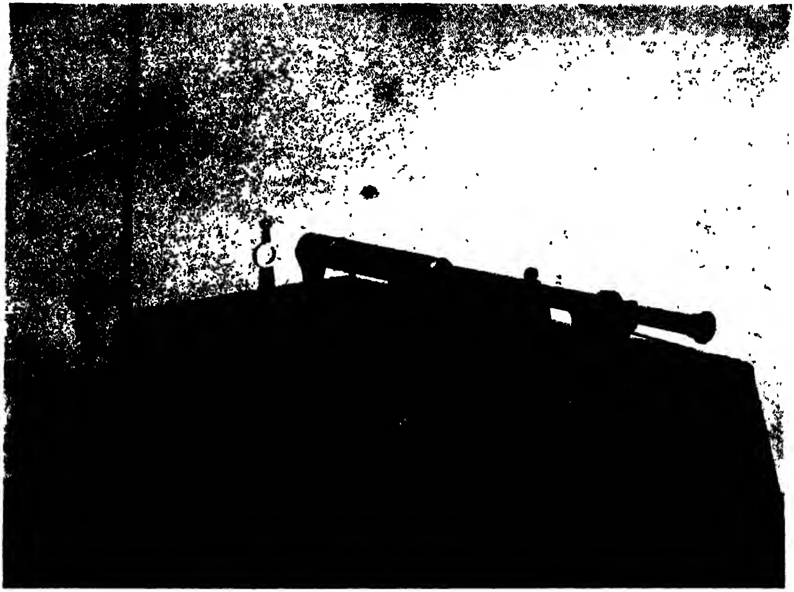

FIG. 22. Box Heliotrope. (*Coast and Geodetic Survey.*)

through the lens to the back surface of the mirror, there to be partly reflected and partly transmitted through the glass. Keeping the tube in this position, the mirror itself must be so turned that the spot of light made visible by this last reflection will appear to cover the hill or station to which the light is to be sent.

One form of heliotrope, in use by the Coast Survey, called a

box heliotrope, consists of a pair of rings with a mirror mounted behind them, and with sights above the rings for pointing. A telescope is sometimes mounted to one side of and parallel to the heliotrope. The various parts remain in position in the box when in use. (Fig. 22.) Some of these heliotropes have no telescope.

The size of mirror used in any heliotrope must be regulated according to the length of line and the atmospheric conditions. Most heliotropes are provided with some arrangement for varying the size of the opening through which the light passes. If the exposed portion of the mirror subtends an angle of about $0''.2$ the amount of light will be sufficient for average conditions. Different atmospheric conditions will require different openings. This matter is of less importance on long lines than it is on short lines.

All heliotropes are provided with a second mirror, usually larger than the first, called the back mirror; this is to be used whenever the angle between the sun and the station is too great to permit sending the ray by a single reflection. The back mirror is set so as to throw light onto the first mirror and the heliotrope is then adjusted to the reflection of the sun as it appears in the back mirror.

16. Lights for Triangulation.

Since 1902 triangulation has been systematically carried on at night as well as during the day. Triangulation at night was found to be somewhat more economical than day observations, because there is almost no delay due to clouds, as is so frequently the case when using heliotropes. Night observations are also more accurate than day observations. The systematic twist in azimuth, which is apt to characterize observations made in sunlight, is absent in the results of night observations.

Acetylene lamps were formerly used for night observations; these have now been superseded by incandescent electric lights. These new signal lamps have a large reflector and are lighted by ordinary dry cells. They are light in weight and easily portable.

Sometimes an ordinary bulb is used, but the best form consists of one having a special filament much reduced in size, so that the light virtually radiates from a point. Some signal lamps are of the automatic type, that is, a clock mechanism switches the light on each night at the hour set for beginning observations and switches it off again at the hour fixed for ending the work for

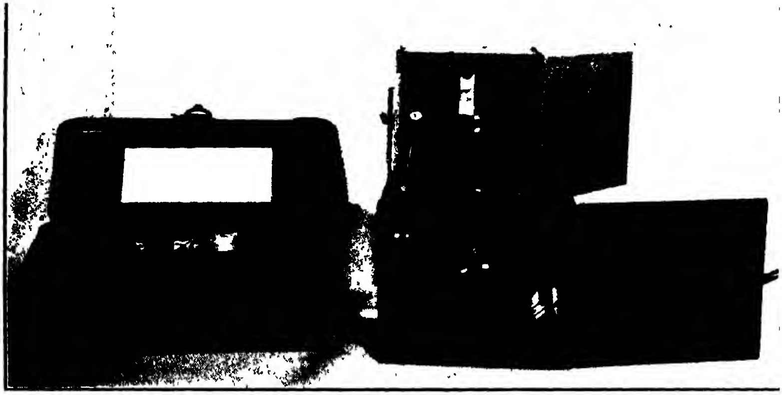

FIG. 23. Signal Lamps for Triangulation. (*Coast and Geodetic Survey.*)

the night. As these clocks will run for several days without winding, one person can attend to several stations; this results in a great saving over the older methods, which required the presence of a light keeper on each station during the entire period of observation. These electric lights are found to be entirely satisfactory on all ordinary triangulation.

17. Observing Towers.

Where a line is obstructed by hills or woods, or where the curvature of the earth is sufficient to make the station invisible, it becomes necessary to construct observing towers. If there is much heavy timber about the station, placing the instrument on the ground may necessitate so much cutting that it will be more economical to erect a tower than to cut the timber.

Observing towers are sometimes made of wood, sometimes of steel. The form of wooden tower which has been used by the United States Coast and Geodetic Survey for many years is very light and slender as compared with the older ones. This kind of tower (Fig. 24) admits of more rapid construction and can be built at a lower cost; it is sufficiently rigid to withstand all

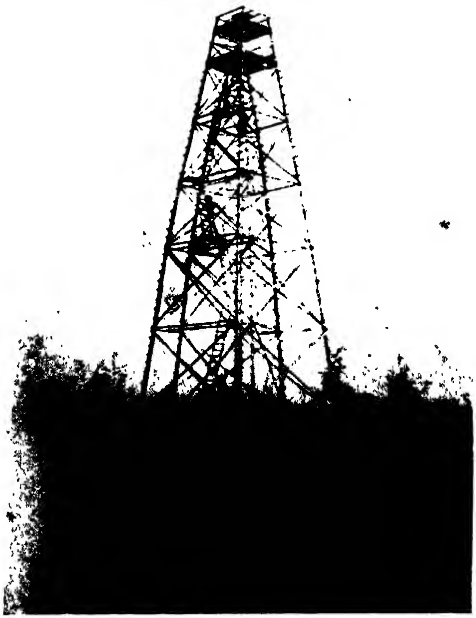

FIG. 24. Eighty-foot Wooden Tower. (*Coast and Geodetic Survey.*)

ordinary storms. The manner of framing the tower is shown in Fig. 25. When the ties are nailed on, the legs are sprung slightly into a bow, thus giving additional stiffness to the structure.

One side of the inner tripod, which is to support the instrument,

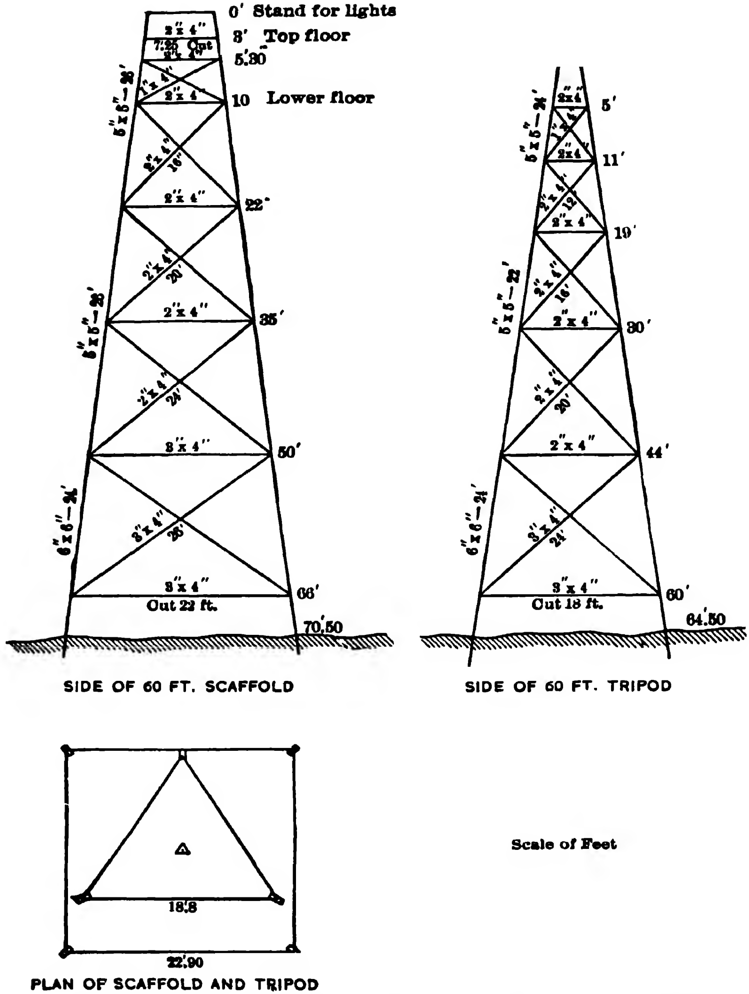

FIG. 25. **Framing Plan of 60-foot Wooden Tower.** (*Coast and Geodetic Survey.*)

is first framed on the ground. This side and the third leg of the tripod are raised into position by a fall and tackle and a derrick, which may be a tree or a section of one of the legs of the outer scaffold. The derrick should be at least two-thirds the height

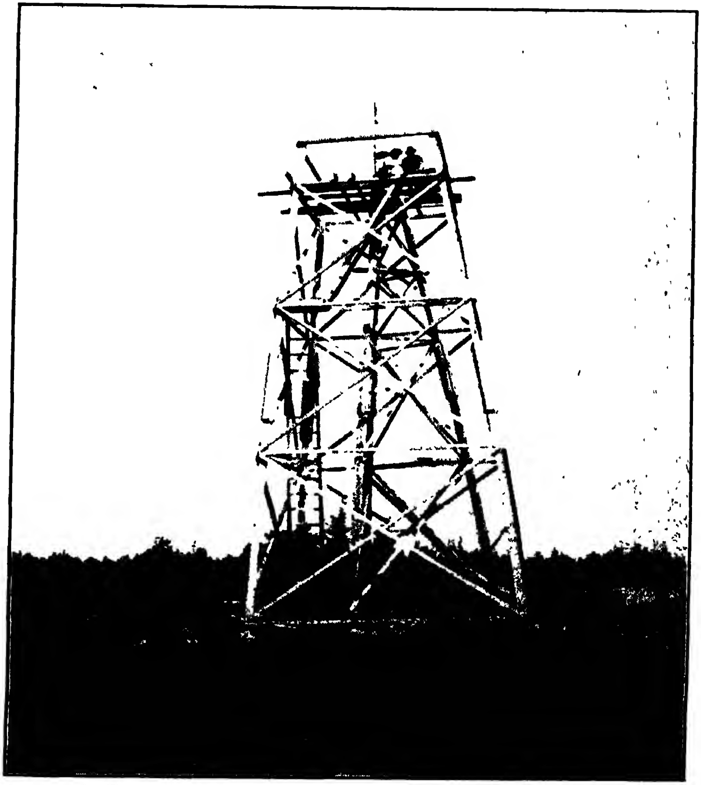

FIG. 26. Forty-foot Wooden Tower.

of the piece to be raised. After the tripod is raised and all braces nailed on, it is itself used as a derrick for hoisting the two opposite frames of the outer scaffold into position. The ties and braces

of the other two sides are then nailed in place. It should be observed that the inner and outer structures are entirely separate, so that the movement of the observer on the platform of the scaffold will not disturb the instrument. The legs of the tripod

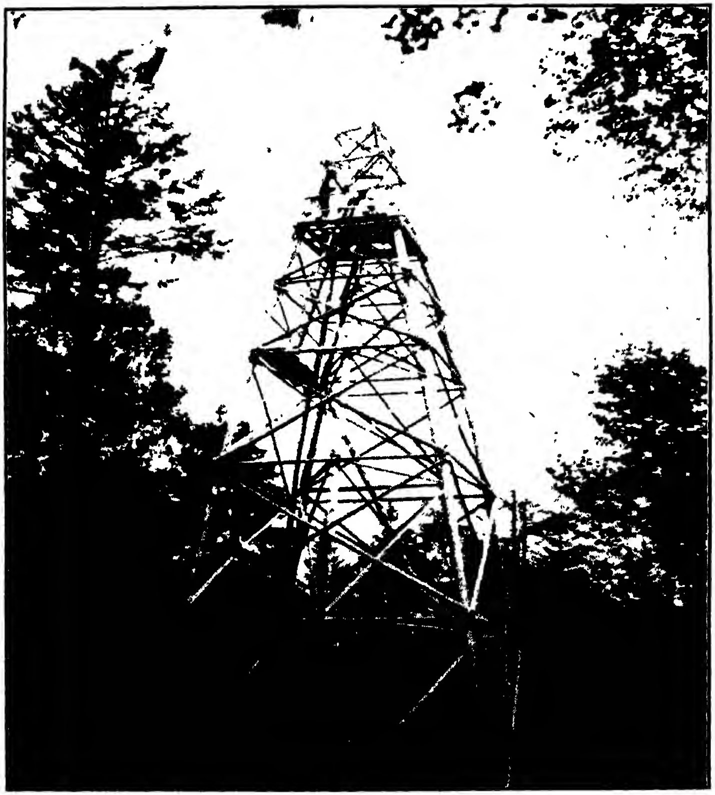

Fig. 27. Fifty-three foot Tower built of poles.

and the scaffold are anchored by nailing them to foot pieces set underground. The outer tower is guyed with wire as a protection against collapse in high winds.

This kind of signal saves lumber, transportation, and cost of construction; it has a small area exposed to the action of the wind; the short ties have the effect of reducing the vibration due to wind, which is troublesome in large towers; the light keeper is placed above the observer (10 ft. or so) and can operate his lights without interfering with the observations. Another advantage of these towers is that the amount of twisting due to the sun's heating is found to be exceedingly small. (For further details consult *Coast Survey Report for* 1903, p. 829.)

In 1927 the Coast Survey tried out a new tower designed by Mr. J. S. Bilby, in charge of signal building on that survey. It is built of steel and is similar in its construction to those used for windmills. Both the inner and the outer towers are tripods. The surface exposed to wind is very small, and the tower is found to be quite steady. As soon as observations are completed at a station the tower may be taken down and shipped to another station, and thus used over and over. This tower can be erected or taken down in a surprisingly short time. The tower is built by placing each rod separately in position and bolting it, instead of hoisting completed frames as in case of the wooden tower.

The United States Lake Survey uses a tower constructed entirely of gas pipe, which has proved to be more economical than timber. It is put together in sections and hoisted as it is built. The upper part of the structure is built first and is then hoisted from the ground by means of tackles; the next section is then added on, all the work being done from the ground. This kind of tower is easy to construct, and the materials portable; the area exposed to wind is very small.

18. Reconnoissance for Base-Line.

With the invar tape apparatus, to be described in detail in Chapter II, base-lines can now be measured over much rougher ground than was formerly possible where bar apparatus had to be used. So far as accuracy goes it is possible to measure a base-line on any grade up to about 10 per cent. Gulleys and

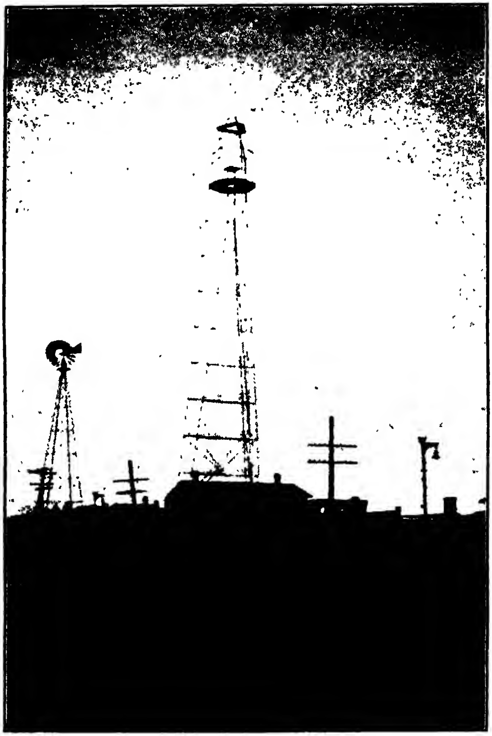

FIG. 27a. Bilby Steel Tower. (*U. S. Coast and Geodetic Survey.*)

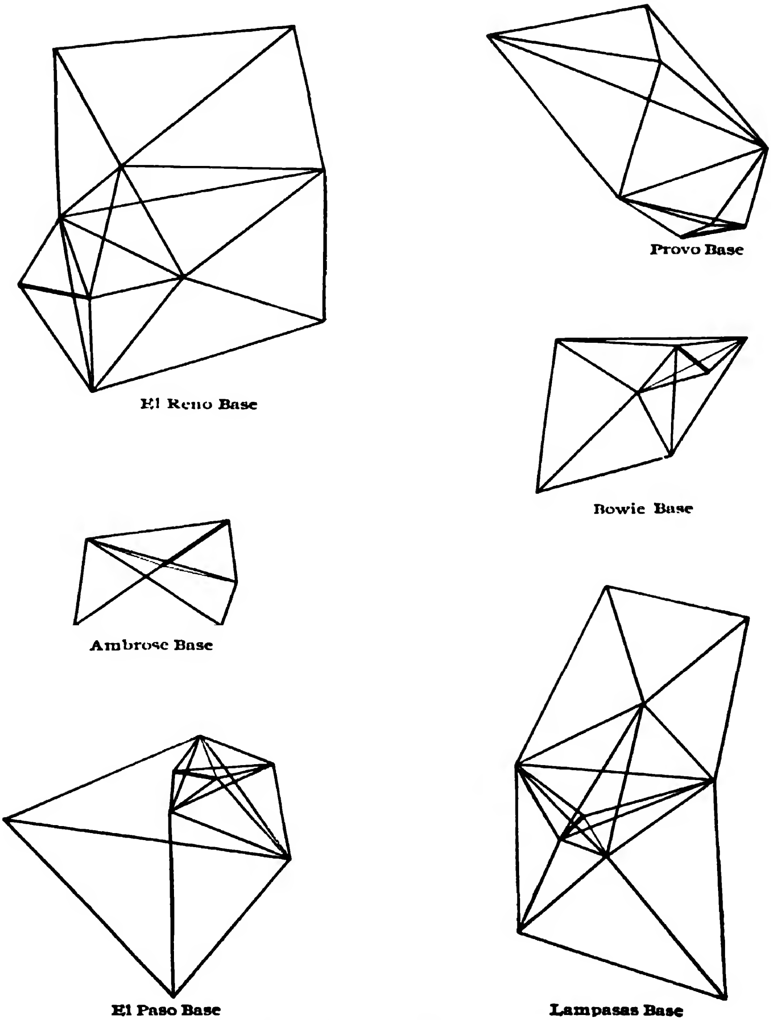

FIG. 27b. Base Nets.

ravines which are not wider than the length of the tape (50 meters) can be spanned without special difficulty. The length of base in first-order triangulation should not be less than 4 kilometers. In the network of triangles connecting the base with the main triangles great care should be used to secure as good geometric conditions as possible; this net should not be longer than two ordinary figures in the triangulation. The strength should be tested in the same manner as for other triangulation. While it is desirable that the base be located on smooth ground, it is always better to place the base on rough ground in order to secure a well-shaped base net, rather than to adopt weak triangles in order to place the base on level ground. In the base net it is permissible to sight over additional lines in order to strengthen the figure. But this should not be carried so far as to make the adjustment difficult and expensive.

Figure 27b shows the base nets on several different belts of triangulation. In some instances it has been found practicable to use the side of one of the main triangles as a base. For example, in the triangulation between Texas and California the Stanton base, which was one of the main lines, (8 miles long) was measured directly with the tape. (Fig. 2a.)

PROBLEMS

Problem 1. What is the strength of the quadrilateral having all the angles equal to 45°? In case one station on the base is not occupied with the instrument, what is the strength? If one station not on the base is unoccupied, what is the strength?

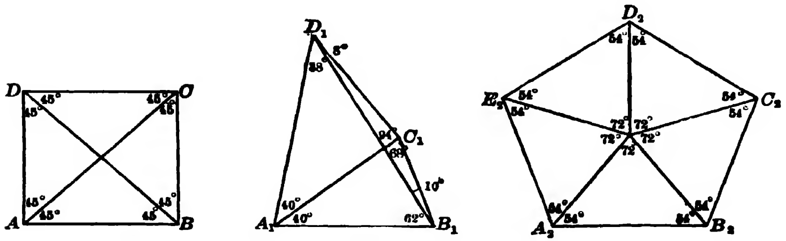

FIG. 28.

Problem 2. Compare the strength of the three figures given in Fig 28.

Problem 3. Compute R_1 and R_2 in Fig. 29; (1) when the interior station is occupied; (2) when the interior station is not occupied.

Problem 4. Three hills A, B, and C are in a straight line. The distance from A to B is 10 miles and the distance from B to C is 15 miles. The elevations are A = 600 ft, B = 550 ft., and C = 650 ft. respectively. Compute the height of a tower to be built on C the top of which will just be visible from A

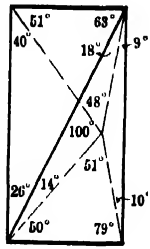

FIG 29.

Problem 5. Four hills A, B, C, and D are in a straight line. The elevations are A = 810 ft., B = 775 ft., C = 1030 ft., D = 1300 ft. respectively. The distances of B, C, and D from A are 8 miles, 28 miles, and 38 miles. Find the height of towers on A and D to sight over B and C with a 10-ft. clearance. The two towers are to be of the same height.

Problem 6. What angle is subtended by a six-inch mast at a distance of twelve miles?

Problem 7. If a fourteen-inch mirror is used on a heliotrope at a distance of 150 miles, what is the apparent angular diameter of the light?

Problem 8. A heliotrope is sighted accurately on a station 20 miles away. How far would one have to go either to the right or to the left of that station before the light from the heliotrope would disappear?

CHAPTER II

BASE-LINES

19. Bar Apparatus for Measuring Bases.

In nearly all the earlier base-line measurements (up to about 1885) the apparatus employed consisted of some arrangement of metal bars. Such apparatus was capable of yielding accurate results, but was slow and cumbersome to use, making the base-line a comparatively expensive part of the survey. An account of the development of base-measuring apparatus will be found in Clarke's *Geodesy* and in Jordan's *Vermessungskunde*, Vol. III; descriptions of numerous forms used in this country will be found in the annual reports of the superintendent of the Coast and Geodetic Survey.

20. Steel Tapes.

Experiments with the use of steel tapes for base-line measurements were made by Jäderin at Stockholm in 1885, by the Missouri River Commission in 1886, and by Woodward on the Coast and Geodetic Survey base at Holton, Indiana, in 1891. The use of steel tapes for this purpose was attended with such success that for twenty years they were very generally used, and by 1900 they had almost wholly superseded the bar apparatus in this country.

The greatest practical difficulty encountered in the use of steel tapes for precise measurement is that of determining the true temperature of the steel when making the measurements in sunlight. The air temperature, as indicated by ordinary mercurial thermometers, is seldom the correct temperature for the tape, except during rainy weather or at night. For this reason it was found necessary to make all measurements of base-lines at night in order to secure the required accuracy.

21. Invar Tapes.

In 1906 the Coast and Geodetic Survey conducted a series of tests on the use of tapes made of the alloy called *invar*. This alloy was discovered by Dr. C. E. Guillaume, Director of the International Bureau of Weights and Measures at Sèvres, near

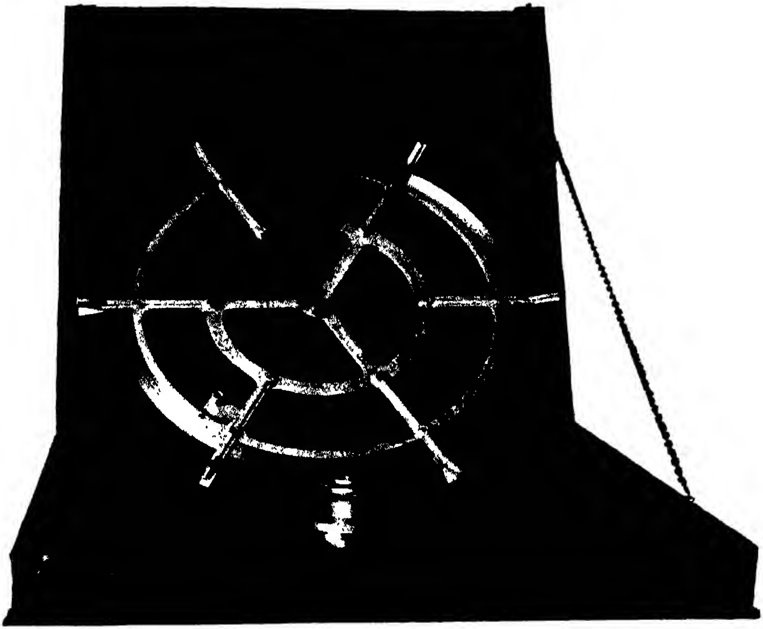

FIG. 30. Invar Tape on Reel.

Paris. It contains about 35 per cent of nickel and 65 per cent of steel, and is given a special heat treatment; this produces a metal having a very low coefficient of expansion. The temperature of a tape during a measurement does not have to be determined so closely as when using steel. Invar is softer than steel and is more easily bent; it must be wound on a reel of not

less than 16 inches diameter to prevent changes in length due to bends, and it must be handled carefully at all times.

Invar tapes were found to give results of the required accuracy for base-lines even when used under the most unfavorable conditions as regards temperature. They have the advantages that the work may proceed in daylight and that it is unnecessary to standardize the tapes in the field, as was done with steel tapes, but the comparison may be made at Washington at the beginning and at the end of the season. Invar tapes proved to be so satisfactory that they have been almost exclusively used for base measurement ever since their introduction.

The coefficients of the earlier tapes were about 0.000 000 3 to 0.000 000 4 per degree centigrade. Tapes were afterward produced having coefficients near to zero, or even negative in sign. Unfortunately the metal in such tapes proved to be unstable, and they are subject to changes in length which are much greater than the changes in the earlier tapes. Efforts are now being made to produce a stable alloy which still has a very low coefficient.

22. Description of Apparatus.

Invar tapes are about 53 meters long with the end graduations 50 meters apart. Intermediate graduations such as every 10 meters or at the 25 meter point, are added if desired. Some tapes have a decimeter at each end sub-divided into millimeters. In section the tape is about 6 mm. by 0.5 mm., and weighs about 25 grams per meter. While in use the tape is supported at three points, at five points, or full length, according to circumstances. An apparatus for setting and holding the zero mark in position is shown in Fig. 31. The bar is set firmly into

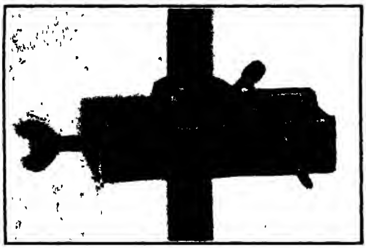

FIG. 31. Apparatus for Setting Zero of Tape

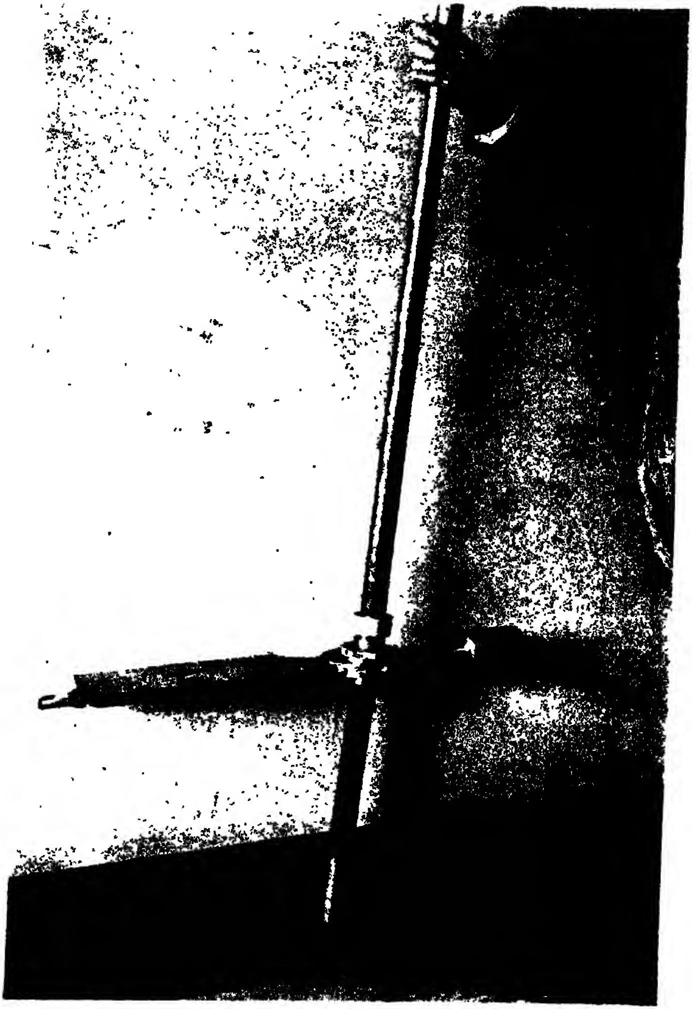

FIG. 32. Tension Apparatus.

the ground and takes the full tension. The ratchet wheel for quick changes in position and the slow-motion screw for the fine adjustment enable the operator to set the zero quickly and accurately against the mark. Such an apparatus is a trifle slower to use than direct setting but permits of great accuracy. The Coast and Geodetic Survey uses a simple bar, the point of which is set into the ground, the setting being made directly by moving the top of the bar. The slow motion adjustment is made by the man who attends to the rear contact; he takes hold of the tape and flexes it sufficiently to bring the zero to the mark.

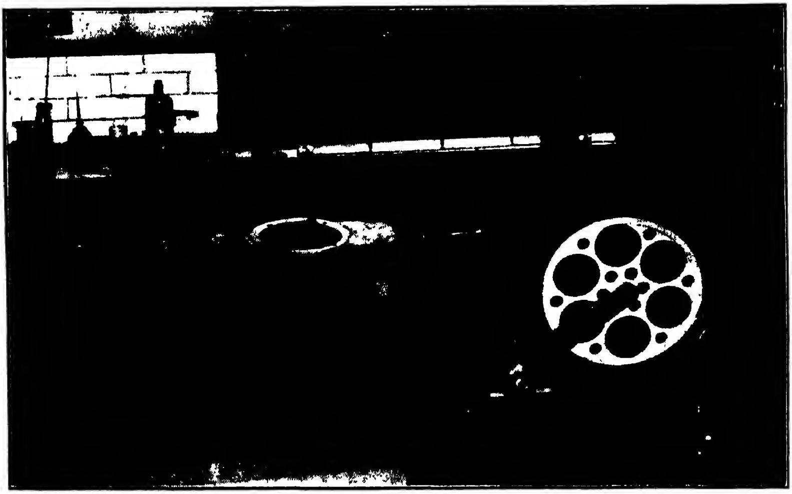

Fig. 33. Pulley for Testing Spring Balance. (*Bureau of Standards.*)

This arrangement permits greater speed than the first and in the hands of trained men yields all the accuracy required. The tension is applied by means of a spring balance the smallest reading of which is 50 grams. The ordinary working tension is 15 kilograms. The stretching apparatus is similar to that shown in Fig. 32. The point of the steel bar is pressed into the ground and

the upper end moved right or left so as to bring the tape over the measuring stake. The balance can be raised or lowered, and clamped in any position by means of a set screw. The reading of the spring balance for a true tension of 15 kilograms is found by means of a standard 15-kilogram weight and a frictionless pulley. (Fig. 33.) The reading of the balance is subject to changes due to temperature; it should be tested at the beginning and at the end of each day's work, and also whenever the load has (accidentally) been released suddenly so as to jar the mechanism of the balance. The thermometers used with this apparatus are graduated to degrees or half degrees and are provided with metal spring clips for fastening them to the tape. It is a good plan to fasten the thermometers to the tape with a piece of adhesive tape, which prevents their changing position or becoming lost. The thermometers are placed one meter inside the end graduations, exactly as they are when the tape is tested in the Comparator. It is important that the thermometers used in the field work should weigh the same and be in the same position as those used in the standardization.

23. Marking the Terminal Points of the Base.

The ends of the base-line are marked in the same manner as triangulation stations. On modern bases the end points are marked by bronze castings set into concrete or into rock. On the older bases they were often marked by copper bolts in drill holes in stone monuments. The points are tied in by reference measurements to aid in the recovery of the station if lost. There is often a sub-surface mark (see Art. 13) to aid in recovering the station if the surface mark is lost or disturbed. Intermediate points on a base are sometimes marked in a similar manner.

24. Preparation for the Measurement.

Before the measuring* can begin the line must be run out and all obstructions removed. This may involve cutting timber, brush, or grass, and leveling off small irregularities in the ground.

* For the methods of the U. S. Coast & Geodetic Survey see Special Publication No. 120, by Major C. V. Hodgson.

Where there are gullies crossing the line, or high obstructions which cannot be leveled off, it may be necessary to build measuring platforms for the observers. Next the measuring stakes, 4 in. × 4 in., are set in position accurately on line and one tape length apart. This must be done accurately enough so that the end marks of the tape will come on top of the stakes. An old (spliced) invar tape, not suitable for the final measurement,

FIG. 33a. Setting the Measuring Stakes.

may be used to advantage in this preliminary work. The stakes must be set firmly and if necessary braced laterally by 1 in. × 4 in. pieces driven into the ground and nailed to the stakes. The stakes must be high enough so that the tape will not touch the ground or bushes at any point. The line is marked in pencil on top of the posts. Before the measurement is begun, strips of zinc or copper are tacked on top of the post so that one edge is exactly on line. All of this line work may be done with an ordinary engineer's transit of good quality. The slope of each tape length as well as the actual elevation of each stake is

then determined by direct leveling on top of the stakes. The absolute elevation is needed for calculating the reduction to sea-level, but this need not be known with great precision. The difference in elevation, however, must be obtained very accurately for the purpose of calculating the grade correction, especially when the grade is steep, say near 10 per cent. Sometimes a clinometer is used for taking the slope when it is important to save time. It is placed either at the middle of the tape or at the middle of one of the loops. The stake for the middle support may be set in advance, like the end stakes, or it may be lined in by eye for both line and grade, when the measurement is made. If the wind is blowing hard it may be necessary to introduce two additional supports at the 12.5 meter and the 37.5 meter marks. When the base-line is on a highway or whenever the use of stakes would be impracticable, iron tripods with small measuring tables on top are used instead of the regular stakes. These tripods have two ball and socket joints for adjusting the height and the position and for leveling the top. If the measuring is done along a railway line the tape is laid on top of a rail. In this case it is advisable to place small rollers under the tape to avoid errors due to friction. Friction on the tops of measuring stakes is a fruitful source of error.

From 3 to 5 kilometers of line can be prepared in a day by a party of 5 men under average conditions.

25. Measuring the Base.

A base may be conveniently measured by a party of six men, the front contact man, rear contact man, front stretcher man, rear stretcher man, middle man, and recorder. The actual measurement is begun by stretching the tape between the terminal point and the first measuring stake. The zero mark of the tape is set over the end mark of the base either by means of a plumb bob or a transit set off line on the perpendicular from the terminal point. The tape must be high enough to clear all obstructions, which may require setting a tripod over the end mark or building a small measuring table. Its height above the

end mark must be measured. The middle support may be lined in by eye with sufficient accuracy, both vertically and horizontally. Square pieces of white cardboard with one corner held at the tape will facilitate lining in the middle point. The rear stretcher man plants his staff in the ground and with the top of his staff behind his shoulder adjusts for height, line and grade, so that the zero mark will fall over the end mark. The forward stretcher man applies the tension gradually until 15 kilograms is reached. Care should be taken not to injure the balance by jerking it or by releasing the tension too suddenly. When the zero is in position and the tension is correct the forward contact man marks with an awl on the copper strip the point opposite the 50-meter mark. The end of the graduation which was used in the comparison is indicated in the certificate; this same end should be used in the field measurements, and the zinc or copper strips must be placed accordingly. The tape should be on top of the stake with its edge in contact with the metal strip, so that their upper surfaces are at the same level. The tape should not drag on top of the stake but should be a millimeter or so above it, so that the contact men have to place a finger on the tape to bring it down in contact with the stake. The thermometers are read as soon as the mark at the 50-meter point is made, and the two temperatures and the distance are recorded. The front contact man reads the front thermometer and the rear contact man reads the rear thermometer. If the readings of tape and temperature are satisfactory, the tape is unhooked from the spring balance and carried forward by the front contact man, the middle man, and the rear stretcher man. It should be held high above the ground so that it will not drag and so that no kinks or twists will occur. Care should be taken not to injure the thermometers. Upon arrival at the next stake the front stretcher man checks by eye the alignment and grade of stakes to see that none has been accidentally moved. The tape is placed in position and the same routine is repeated. The rear contact man attends to setting the zero line on the

scratch already made, using an ordinary reading lens for this purpose. The front contact man is usually in the best position to judge when all conditions are satisfactory for making the readings, and he usually acts as chief of party.

Any short distance, at the end of a base, for example, may be measured with a section of the invar tape or with a metric steel tape. Whenever the spacing of the stakes is such that the 50-meter mark will not fall on top of the stake a "set-up" or a "set-back" may be made on the preceding stake. This consists in measuring forward or backward a few centimeters and making a new scratch from which to measure. Careful record should be made so that no mistake will occur in computing the final length. The metal strips are usually preserved as a part of the record.

By the process just described a base-line can be measured at the rate of about 2 kilometers an hour by a party of six men.

It has been found that when measurements are made in a rain or in a heavy fog the weight of the water which adheres to the tape is sufficient to introduce more error than can be tolerated in the best grade of base measurement. Therefore no measurements of a first-order base should be made when it is evident that an error will enter from this source.

26. Accuracy Required.

The base-line is divided into sections about 1 kilometer in length; it is also divided into 3 divisions, the end of a division coinciding with the end of a section. At least 3 different tapes are used in the measurement of the same base, and these are used in such a way as to give an intercomparison of the tapes and detect any possible change of length. A different pairing of tapes is used on each section of the three divisions. Each tape will be run forward on one division and backward on the other. Only two measurements of any section are made unless the discrepancy in millimeters between the two measurements exceeds $10\sqrt{K}$ (K being the length of the section in kilometers), in which case additional measurements are made until two meas-

urements do agree within this limit. The two complete measurements of the base should give an accuracy for the result represented by a probable error not greater than one in one million. The actual errors due to poor alignment, marking the tape lengths, and errors due to grade, tension, and temperature should not individually exceed one part in half a million.

27. Computation of Probable Error of Base.

In the Report of the Superintendent of the U. S. Coast and Geodetic Survey for 1910 there is given a method for calculating the probable error (p.e.) of a base. Three causes of error were considered: 1. the uncertainty in the lengths of the tape; 2. the error in the temperature coefficient, and 3. the errors of measurement. In computing the p.e. of each section the error in length was taken as the number of lengths in the section (n) times $\frac{1}{2}\sqrt{e_1^2 + e_2^2}$, where e_1 and e_2 are the errors of the two tapes used in the measurement. The error due to temperature coefficient was taken as $n\,(t_0 - t) \times \frac{1}{2}\sqrt{c_1^2 + c_2^2}$, in which n is the number of tape lengths, t_0 the standardization temperature, t the observed temperature, and c_1 and c_2 the p.e.'s of the coefficients of the two tapes. The p.e. of a section due to errors of measurement is $r_0 = .6745\sqrt{\dfrac{\Sigma v^2}{n'\,(n' - 1)}}$ in which v is a residual, and n' is the number of measures of a section. The final p.e. of a section is the square root of the sum of the squares of these three errors. The p.e. of the entire base is the square root of the sum of the squares of the probable errors of the sections.

It is often considered that the errors of standardization and determination of coefficient are included in or are masked by discrepancies in the measured lengths of the sections. The probable error of each section is therefore computed by the equation

$$\text{p.e.} = 0.6745\sqrt{\frac{\Sigma v^2}{n(n - 1)}}$$

where v is a residual and n the number of measures of this section. Where a section is measured but twice the p.e. will

be 0.6745 times one-half the difference between the two measured lengths. The p.e. of the whole base is the square root of the sum of the squares of the probable errors of the sections.

28. Corrections to Base-Line Measurements. - Correction for Grade.

Where the slope is determined by direct leveling, the most convenient formula for computing the horizontal distance is one involving the difference in elevation of the ends of the tape. In Fig. 34, let h be the difference in elevation of the end points

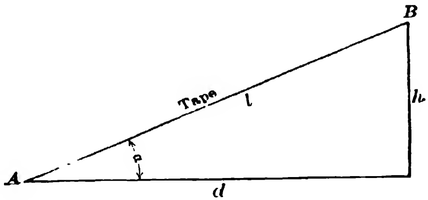

FIG. 34.

A and B, and let l be the length and d the required horizontal distance. Then

$$\text{Corr. for grade} = C_g = l - d = l - \sqrt{l^2 - h^2} = l - l\sqrt{1 - \frac{h^2}{l^2}}.$$

But
$$\left(1 - \frac{h^2}{l^2}\right)^{\frac{1}{2}} = 1 - \frac{h^2}{2\,l^2} - \frac{h^4}{8\,l^4} - \cdots .$$

Therefore
$$C_g = l - l\left(1 - \frac{h^2}{2\,l^2} - \frac{h^4}{8\,l^4}\cdots\right)$$
$$= \frac{h^2}{2\,l} + \frac{h^4}{8\,l^3} + \qquad [1]$$

If the slope has been found in terms of the vertical angle α, the correction may be computed by the expression

$$C_g = 2\,l\sin^2 \tfrac{1}{2}\,\alpha = l \text{ vers } \alpha. \qquad [2]$$

In good base-line work the errors in length due to errors in determining the grade should never exceed one part in half a million.

29. Correction for Alignment.

The errors in aligning a base-line can easily be kept so small as to be negligible. If any point is found, however, to be out of line by an amount sufficient to affect the length, the correction may be computed by formula [1].

30. Correction for Temperature.

The temperature correction may be computed if we know the coefficient of expansion, the actual temperature of the tape and the standard temperature, and the measured length of line. If k is the coefficient, t the observed temperature, t_0 the standard temperature, and L the measured length, then

$$\text{Temperature correction} = +kL(t - t_0). \qquad [4]$$

The temperature correction is often expressed as a term in the tape equation, as shown in the following article.

30a. Correction for Absolute Length.

The length of the tape is often expressed in the form of an equation, such as

$$T_{516} = 50^m + (12.382^{mm} \pm 0.016^{mm}) + (0.0178^{mm} \pm 0.0007^{mm})\,(t - 25^\circ.8\text{ C.}), \qquad [5]$$

meaning that tape number 516 is 12.382^{mm} more than 50^m long at a temperature of $25^\circ.8$ C., and that 0.016^{mm} is the uncertainty of this determination. The quantity 0.0178 is the change in length of the 50^m tape for a change in temperature of 1°, and 0.0007 is the uncertainty in this number. (The temperature coefficient for this tape is 0.000,000,356.)

According to the present practice, tapes are standardized at the Bureau of Standards, Washington, under exactly the same conditions, in regard to tension, temperature determination, and manner of support, as those which are to govern the field measurements. By this means all uncertainty in the absolute length and in the tension correction is kept within narrow limits.

31. Reduction of Base to Sea-Level.

In order that all triangulation lines may be referred to the

same surface it is customary to employ the length of the line at sea-level between the verticals through the stations.

In Fig. 35 let B represent the measured base at elevation h above sea-level (supposed spherical), and b the length of base reduced to sea-level, R_α being the radius of curvature of the surface (see Art. 97 and Table XI). Then, since the arcs are proportional to their radii,

$$\frac{b}{B} = \frac{R_\alpha}{R_\alpha + h}$$

and

$$b = B \cdot$$

$$+ \frac{\cdots}{R_\alpha}$$

$$= B\left(1 - \frac{h}{R_\alpha} + \frac{h^2}{R_\alpha^{\,2}} - \cdot \cdot \cdot\right).*$$

Therefore the reduction to sea-level is

$$b - B = -B\frac{h}{R_\alpha} + B\frac{h^2}{R_\alpha^{\,2}} - \cdot \cdot \cdot \cdot \quad [6]$$

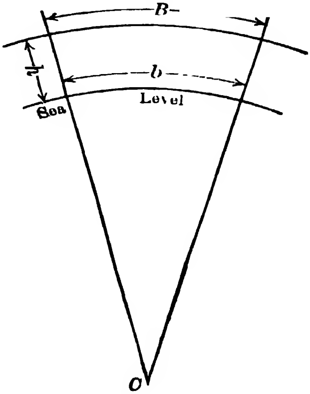

Fig. 35.

Each section is ordinarily reduced separately, unless the whole base is at nearly the same elevation. The elevation used is the means of the elevations of the ends of the sections.

Question. Is it necessary to reduce each triangulation line separately to sea-level?

32. Correction for Sag.

Between any two consecutive points of support the tape hangs in a curve known as the *catenary*, its form depending upon the weight of the tape, the tension applied, and the distance between the points of support.

In Fig. 36 let l be the horizontal distance between the supports, the two being supposed at the same level; let n be the number of such spans in the tape-length, t the tension, and w the weight of a piece of tape of unit length. Also let v equal the (vertical)

* See formula at foot of page 408.

sag of the middle point of the tape below the points of support. Since the curve is really quite flat under the tension actually employed in field-work, the length of the catenary will be sensibly equal to that of a parabola whose axis is vertical and which

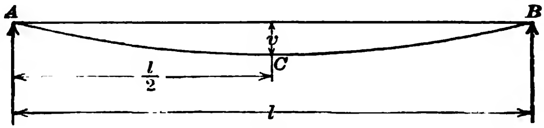

FIG. 36.

passes through the points A, B, and C. The equation of this parabola is $x^2 = \frac{l^2}{4v} \cdot y$, and the length of curve, found by the usual method of the calculus, is $2s = l + \frac{8v^2}{3l} + \cdots$. The difference, $2s - l$, between the length of curve AB and the chord* AB is approximately

$$2s - l = \frac{8}{3} \times \frac{v^2}{l}. \qquad [a]$$

If we consider the forces acting on the tape at the point C, and take moments about the point of support A, we have

$$\frac{wl}{2} \times \frac{l}{4} = v \cdot t.$$

Therefore

$$v = \frac{wl^2}{8t}. \qquad [b]$$

Substituting in [a] the value of v found in [b], we find that the shortening of this section of tape due to sag is

$$2s - l = \frac{8}{3l}\left(\frac{wl^2}{8t}\right)^2 = \frac{l}{24}\left(\frac{wl}{t}\right)^2.$$

For n sections, we have $nl = L$, whence

$$\text{Correction for sag} = C_s = \frac{L}{24}\left(\frac{wl}{t}\right)^2. \qquad [7]$$

* If we assume that the curve is circular the same length is obtained if we drop terms of the 5th power in the series.

33. Tension.

The modulus of elasticity of the tape due to the tension applied equals the stress divided by the strain. If a = the elongation and L the length, and if t equals the tension and S the area of the cross-section, then the modulus of elasticity E is given by

$$E = \frac{t}{S} \div \frac{a}{L}$$
$$= \frac{Lt}{Sa}.$$

The elongation is

$$a = C_t = \frac{Lt}{SE}, \qquad [8]$$

where C_t is the correction for the increase in length due to tension. Evidently the difference in length due to a change from tension t_0 to tension t is $a = \frac{L}{SE}(t - t_0)$.

The value of E must be found by trial, applying known tensions and observing a directly.

To allow for slight variations in tension, such as those due to the failure of the spring balance to give the desired reading the instant the scale of the tape is read, the correction may be derived as follows:

Since the effective length of the tape depends both upon the elongation due to tension and upon the shortening due to sag, and since these both involve t, the variation may be found by differentiating the expression

$$L_1 = L + C_t - C_s$$
$$= L + \frac{L \cdot t}{SE} - \frac{L}{24}\left(\frac{wl}{t}\right)^2,$$

regarding t as the independent variable. The differentiation gives

$$dL = \frac{L}{SE}dt + \frac{L}{12} \cdot \left(\frac{wl}{t}\right)^2 \cdot \frac{dt}{t} \qquad [9]$$

This is the correction due to small variations in t. This quantity may be found more satisfactorily by actual tests, varying t by known amounts and observing the change in length directly.

It was once the practice to compare the tape with the standard when it was supported its entire length, and to calculate the sag and tension corrections to obtain the effective length when supported at a few points. The present practice of comparing the tape under the same conditions that are to exist in the field-work eliminates all uncertainty in these computed corrections.

33a. Change in Weight or Position of Thermometers.

If the thermometers used are of different weight from those used in the standardization, or if placed in a different position on the tape it is possible to compute the effect on the length by means of a formula given by Dr. W. D. Lambert of the Coast and Geodetic Survey. If s is the length of the curve (catenary) with the thermometer attached, s_0 is the length without the thermometer, a is half the distance between supports, h is the distance from the center of the curve to the thermometer, $c = T \div m$, where T is the tension and m the weight of tape per unit of length, and $l = p \div m$ where p is the weight of the thermometer, then

$$- s_0 = \left(\frac{a^2 - h^2}{2c^2}\right)\left(l + \frac{l^2}{2a}\right).$$

This gives the change in length for a single loop. If the tape is supported in the middle the result should be doubled. If we assume a 50-meter tape weighing 25 grams per meter, supported at the middle, having two thermometers each weighing 25 grams attached one meter from the ends, and under a tension of 15 kilograms, then $s - s_0 = 0.000\,0347$ meter. If we change p to 45 grams then $s - s_0 = 0.000\,0643$ meter. Therefore a change in the weight of the thermometers from 25 to 45 grams changes the length of the tape 0.000 0592 meter, or more than 1 part in 1 million.

34. Standardization of Tape.

The standard of length for the United States is the meter deposited with the International Bureau of Weights and Measures at Sèvres, near Paris. This is a platinum-iridium bar with three fine lines at each end. The distance between the middle lines of the two groups, when the bar is at temperature 0° C., and is supported at the neutral points 28.5 cm. each side of the center, is one meter by definition. Two copies of this bar (called prototype meters) Numbers 27 and 21 are in possession of the United States and are deposited with the U. S. Bureau of Standards at Washington.*

These were received from the International Bureau of Weights and Measures in 1889. These meters are composed of an alloy of 90 per cent platinum and 10 per cent iridium, and are X-shaped in cross-section.

Bar M 27 is the standard of length in the United States. Its length equation is

$$\text{No. } 27 = 1\ m - 1.6\ \mu + 8.657\ \mu t + 0.00100\ \mu t^2$$

where 1 micron (μ) = 0.000,001 meter and t is the temperature in centigrade degrees of the international hydrogen thermometer. This bar was compared with the Paris meter in 1904 and again in 1922. No appreciable change was detected.†

The probable error of the comparison of the national prototypes with the international meter was found to be $\pm 0.04\ \mu$. It is estimated that the uncertainty in the lengths at temperatures between 20° and 25° lies between $\pm 0.1\ \mu$ and $\pm 0.2\ \mu$.

Bar M 27 is preserved in a fireproof vault at the Bureau of Standards and is used only for comparison with No. 21 and other platinum-iridium bars. No. 21 is used for comparison with secondary standards, and for testing tapes for geodetic work.

When a 50-meter invar tape is to be tested, the first step is to

* For details see Circular No. 332, Bureau of Standards, 1927.

† See "Creation du Bureau International de Poids et Mesures." — by Ch.- Éd. Guillaume, Director of the Bureau. Paris, 1927.

test the 50-meter base in the tape testing tunnel. This distance is marked by means of two spheres set into concrete posts so that the tops of the spheres are at the level of the floor. The measurement from the end sphere to the first microscope (also on a concrete post) is made by the use of what is known as the "cut-off cylinder." (Fig. 37.) This consists of a vertical tube having at the lower end a conical hole, the axis of which coincides with the axis of the cylinder. When the cylinder is placed over the sphere the center of the sphere is always exactly in the axis of the cylinder. A spirit level is provided for making the axis of the cylinder truly vertical. On top of the cylinder is a millimeter scale on which readings can be taken with the microscope.

At every 5 meters* in the tunnel is a post for supporting a microscope. Between the 8th and 9th posts are additional posts set every meter. The 50-meter distance is tested by means of a steel 5-meter bar known as the Woodward bar, or "B 17." (Fig. 38.) This steel bar is compared with No. 21 at intervals by means of the microscopes set 1 meter apart. When in use B 17† is packed in melting ice in a Y-shaped trough to control the temperature.

The complete test of the tape consists in measuring first a 5-meter distance between microscopes by means of M 21, then a 50-meter distance between the spheres by means of B 17, and finally a comparison of the tape with the 50-meter distance, the tape being supported in the same manner and under the same tension that it is in the field measurements.

The error in laying off the 5-meter distance has been estimated as 1.1 μ, or about 1 in 5 000 000, and that of laying off the 50-meter distance (in terms of 10 times the length of B 17) as 0.015 mm. The error of comparing the tape with the base is

* See Circular No. 328, Bureau of Standards, 1927, by Dr. Lewis V. Judson, associate physicist.

† See Appendix 8, U. S. Coast and Geodetic Survey Report for 1892, p. 339. On p. 480 of the same report will be found an article on the theory of tapes.

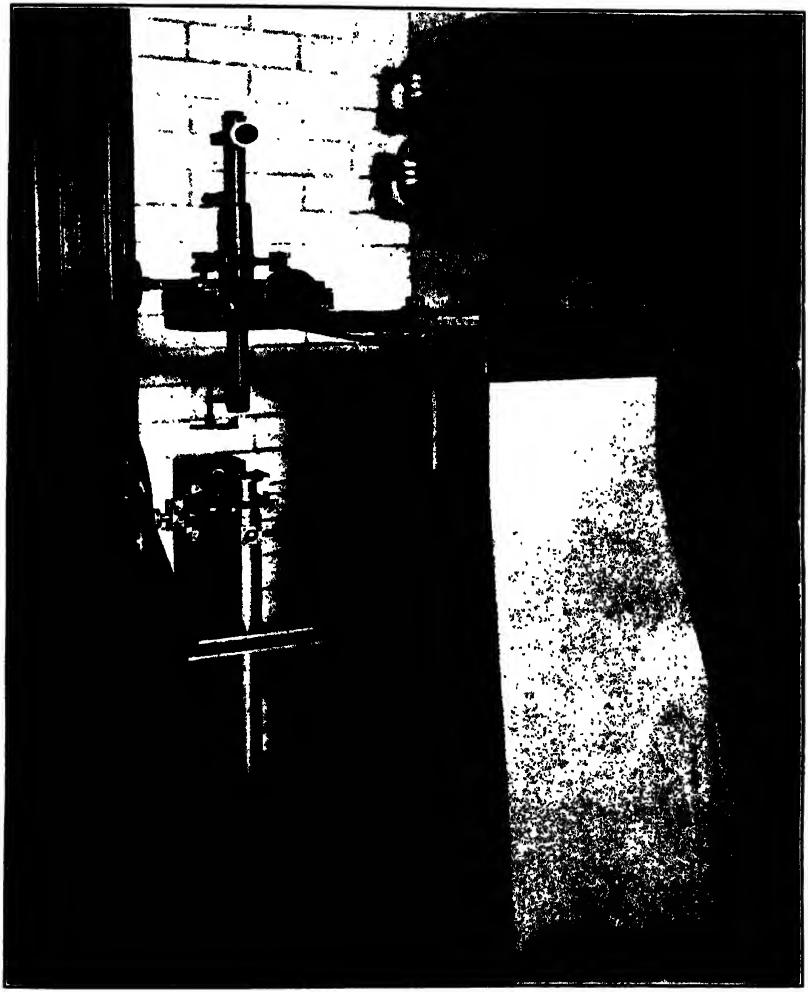

FIG. 37. The Cut-off Cylinder. (*Bureau of Standards.*)

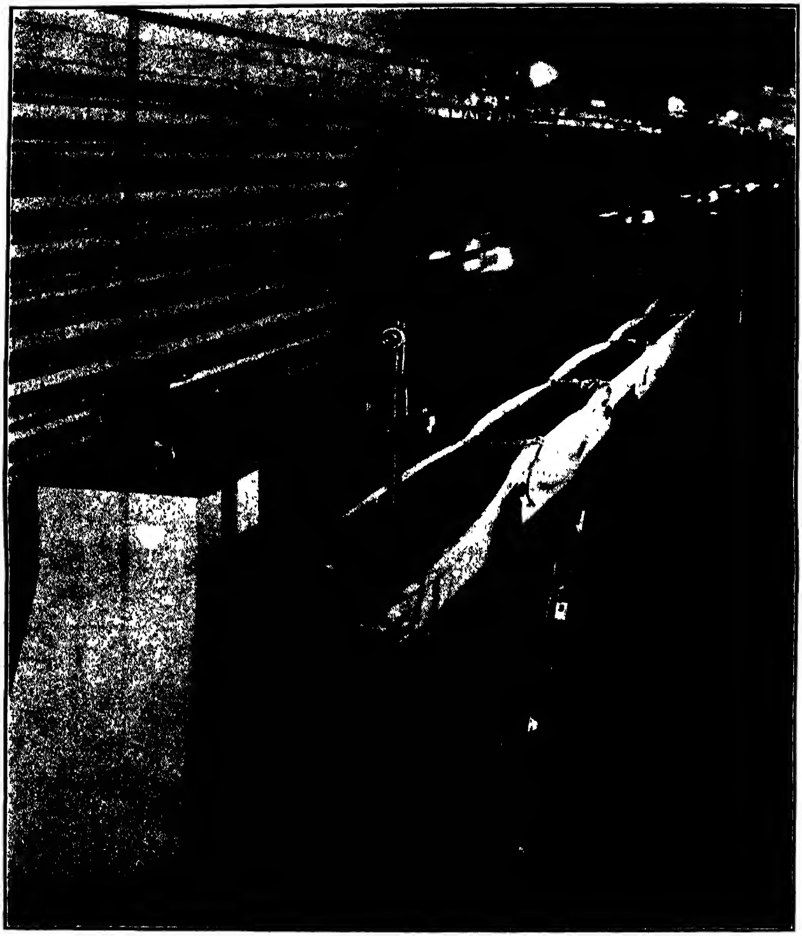

FIG. 38. Tape Testing Tunnel at Bureau of Standards. (Showing 5-meter steel bar in Y-shaped Trough.)

LVJ:JPM
II-1

DEPARTMENT OF COMMERCE

Bureau of Standards

CERTIFICATE

FOR

Bureau File Reference:
II-1, Test No. Twl 51915

50-METER INVAR TAPE

B. S. No. 4111

Maker: *No. S.I.P. 215*

SUBMITTED BY

Mass. Institute of Technology,
Cambridge, Mass.

THIS CERTIFIES that the above-described tape has been compared with the standards of the United States and found to have the length given below when under a horizontal tension of 15 kilograms and when supported at the 0, 25, and 50 meter points:

(0 to 50 meters) = *49.99639* **meters, at** *21.4* **°C.**

When supported at the 0, 12.5, 25.0, 37.5, and 50 meter points:

(0 to 50 meters) = *49.99908* **meters, at** *21.4* **°C.**

When supported on a horizontal surface, throughout its entire length: *(value computed from observations taken on the tape when supported at 3 and at 5 points).*

(0 to 50 meters) = *50.00004* **meters, at** *21.4* **°C.**

For the first and second of the above conditions thermometers weighing *45* **grams each were attached at the points 1 meter inside the terminal marks during the test.**

The above comparisons were made on the section of the lines near the end on the edge of the tape *farthest away from the observer when the zero is at his left.*

The above values are correct within 1 part in *300,000*; **the probable error does not exceed 1 part in** *1,000,000.*

The thermal expansion coefficient of this tape was found to be ($t°$ − 21.4 °C.) (0.017 mm. ± 0.001 mm.) per 50 meters.

The weight per meter of this tape was found to be 24.7 grams.

Test completed, *February 11, 1928*

L. J BRIGGS, *Acting Director*
GEORGE K. BURGESS, *Director.*

Washington, D. C.

Form 584

from 1 in 10 000 000 to 1 in 5 000 000. It is believed that the absolute error of the determination of the tape length does not exceed one part in one million. The certificates furnished with the tapes state that the probable error of the length given does not exceed one part in 1,500,000 and that the absolute error does not exceed 1 in 500,000. The principal source of error in the determination of the tape length is apparently the uncertainty in the length of the 1-meter bar itself, although this error is so small that it cannot affect the tape standardizations appreciably. In other words, the least accurate determination in the standardization process is more accurate than is ever required for geodetic work.

PROBLEMS

Problem 1. Derive the equation of the parabola mentioned in Art. 32. Compute the length of the parabola between the points of support A and B.

Problem 2. The difference in elevation of the ends of a 50-meter tape is 7.22 ft., obtained by leveling. What is the correction (in meters) to reduce the slope distance to horizontal distance?

Problem 3. The length of a base-line is $17486^m.5800$, measured at an (average) elevation of 34.16 meters above sea-level The latitude of the middle point is 38° 36′. The azimuth of the base is 16° 54′. What is the length of the base reduced to sea-level?

CHAPTER III

FIELD-WORK OF TRIANGULATION — MEASUREMENT OF HORIZONTAL ANGLES

35. Instruments Used in measuring Horizontal Angles. Instruments intended for triangulation work are of two kinds: the *direction instrument*, first designed in England by Ramsden in 1787; and the *repeating instrument*, first used in France about 1790. The former is the one chiefly used at the present time for triangulation of the highest order; the repeating instrument, on account of its comparative lightness and simplicity, is much used on triangulation of less importance.

Triangulation instruments are larger than ordinary surveying transits, the diameter of the horizontal circles varying from 30 inches in very old instruments to about 8 inches in modern instruments. It has been found by experience that small circles can be graduated so accurately that nothing is gained by using very large circles, the accuracy of results depending more upon the accuracy of the dividing engine than upon the size of the graduated circle itself. Furthermore, the smaller circles are less affected by flexure than the larger circles and are much easier to transport. All triangulation instruments are designed with three leveling screws; this produces a much more stable instrument than the four-screw transit. The instrument is usually mounted on a solid support, such as a concrete pier, a wooden stand, or on the tripod of an observing tower. Such instruments are not used on ordinary tripods for precise work.

36. The Repeating Instrument. The repeating instrument has an upper and a lower plate arranged exactly as in the surveyor's transit, and the graduated circle is read by two or more verniers graduated to 10″

or to 5″. Verniers reading finer than 5″ are not practicable, and dependence must be placed upon the repetition principle

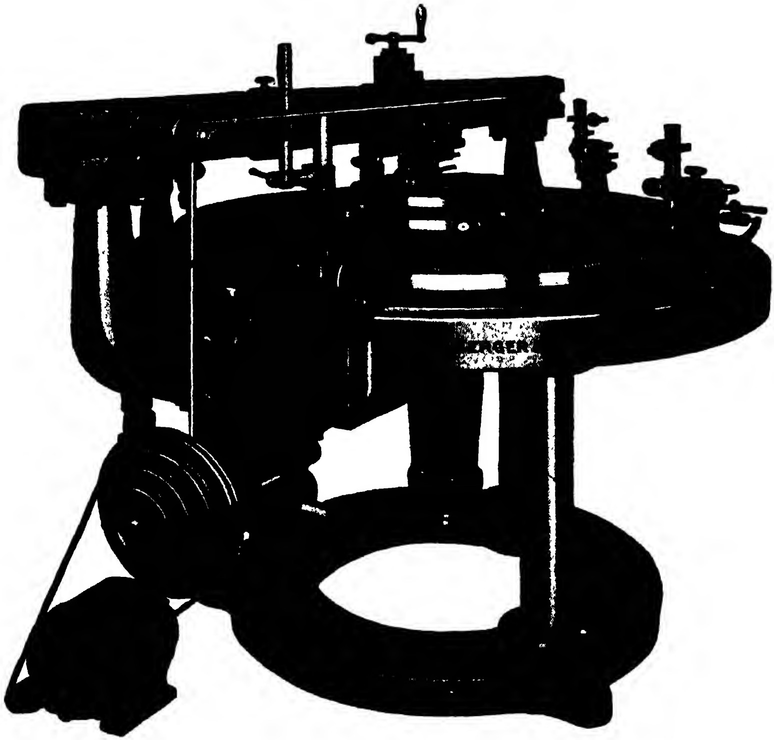

FIG. 39. Dividing Engine for Graduating Circles. *(C. L. Berger & Sons.)*

for securing greater precision. Figure 40 shows a repeating instrument having an 8-inch circle which is read by two opposite verniers to 10 seconds. The telescope of this instrument has an aperture of 1½ inches and a magnifying power of 30. Since an instrument of this kind is likely to be used in sighting on pole signals, the cross-hairs are usually arranged in the form of an

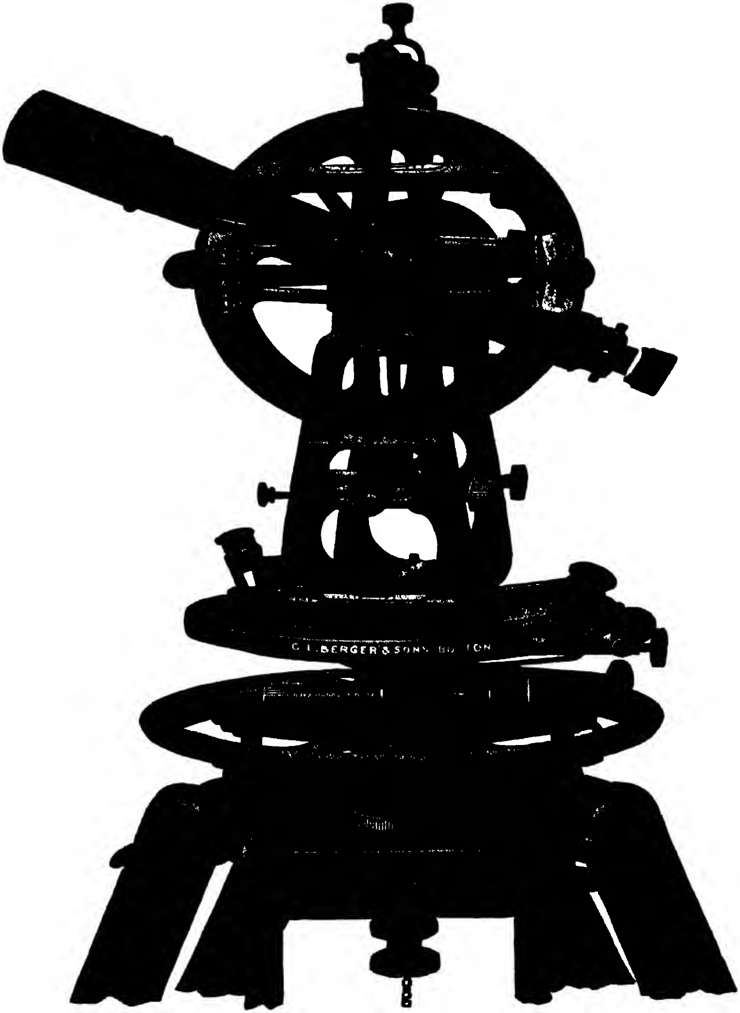

FIG. 40. Repeating Instrument. (*C. L. Berger & Sons.*)

X, the pole being made to bisect the angle between the hairs when the pointing is made. Single vertical hairs would not be practicable except on very short lines and wide signals, as the width of the ordinary hair is so great that it would completely obscure the pole on long distances.

37. The Direction Instrument.

The direction instrument has but one horizontal circle, read by two or more micrometer microscopes instead of verniers. The circle can be turned freely about the vertical axis so as to make any desired setting in the microscopes. In some instruments the circle is clamped in position, while in others it is held by friction alone. The peculiarity of the direction instrument is that the motion of the alidade, that is, the upper part carrying the telescope and the microscopes, is entirely independent of the motion of the circle itself; the latter can be shifted while the upper part remains clamped. This would be impossible, of course, with an ordinary transit. It is evident that a repeater could be used as a direction instrument, but that a direction instrument could not be used for measuring angles by the repetition method.

The circle of the direction instrument is usually graduated into 5′ spaces. The direction of the line of sight of the telescope, referred to the direction of the 0° graduation, is read by first noting the degrees and 5′ spaces in the microscope chosen as the index microscope, and then accurately measuring with each microscope the fractional part of the 5′ space which lies between the zero point of that microscope and the last preceding graduation. The micrometers can usually be read to seconds directly, and to tenths of seconds by estimation. The mean of the results from all the microscopes added to the reading of the degrees and 5′ spaces is taken as the direction of the line of sight of the telescope.

The telescope of the 12-inch Coast Survey theodolite (Fig. 41) has an aperture of 2.4 inches, a focal length of 29 inches, and magnifying powers of 30, 45 and 60. The circle is graduated to

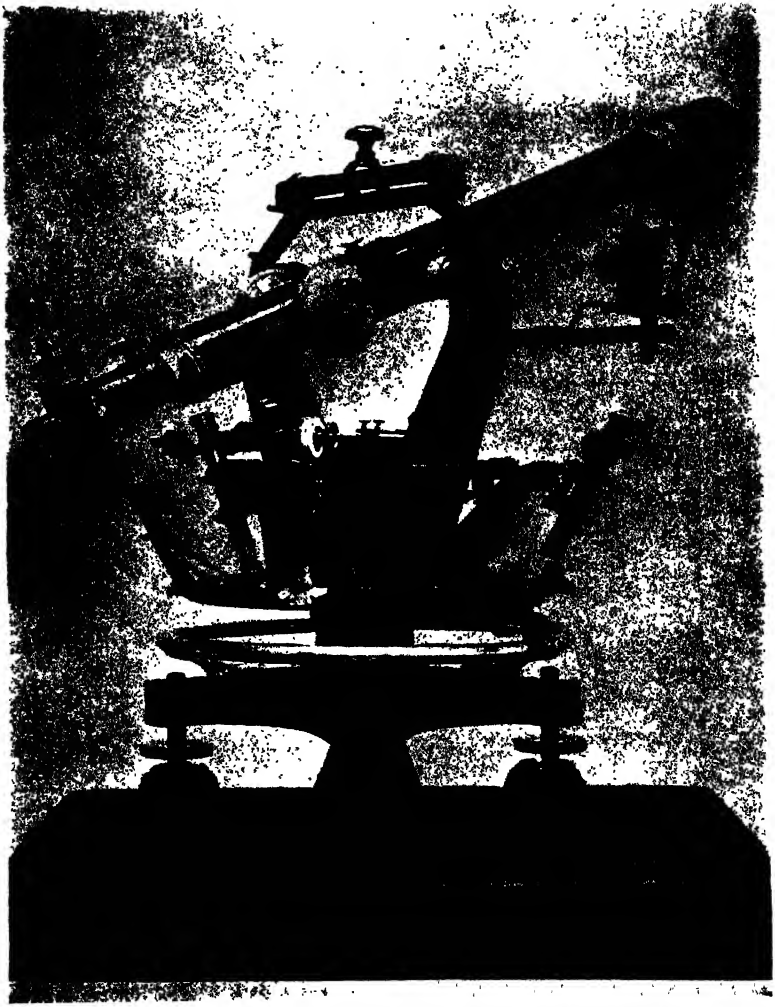

FIG. 41. Twelve-inch Theodolite. (*Coast and Geodetic Survey.*)

5′ and is read to single seconds by means of three microscopes. A small index microscope is provided for reading the degrees and 5′ spaces. A camel's-hair brush (inside the cover plate)

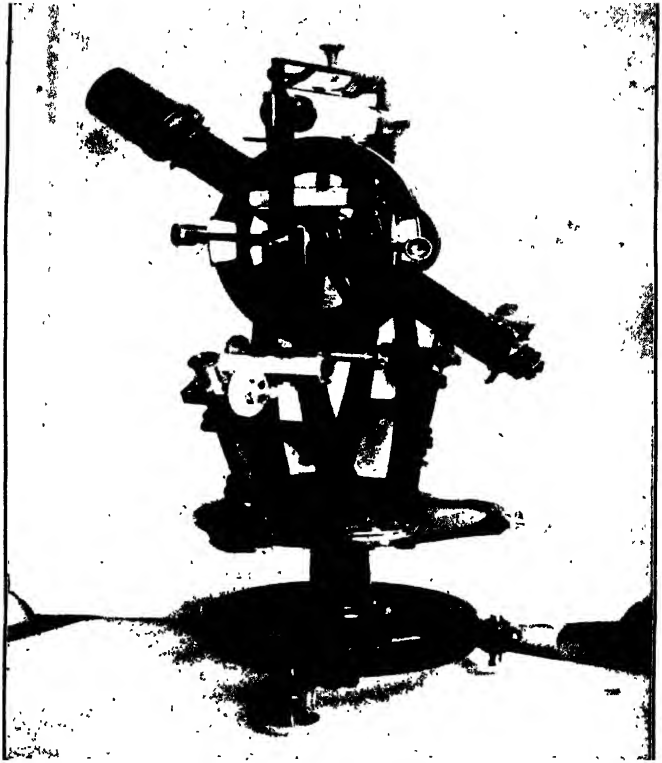

FIG. 42. Hildebrand Theodolite.

sweeps over the graduations. The base is made very heavy, and the bearing surfaces of the centers are glass hard. The centers on this instrument are very long. The upper parts of the instrument are made chiefly of aluminum, in order to dimin-

ish the weight bearing upon the centers and thus reduce wear. This design produces an instrument of exceptional stability. In some of these instruments the micrometers are provided with two sets of parallel hairs set about 4′ apart. When a setting is made on a graduation to the left the left-hand pair is used; when setting on the next graduation to the right the right-hand pair is used. (See Art. 39.) This saves turning the screw through five whole revolutions each time the microscope is read.

In some instruments the microscopes are vertical and are read by means of angle prisms. This makes the whole instrument more compact.

The Hildebrand theodolite (Fig. 42) has a circle about 21 centimeters in diameter and is read by two microscopes which read directly to single seconds. The focal length is 380 mm. and the magnifying powers are 42 and 56. The microscopes are marked A and B; the A microscope is ordinarily used as the index microscope, i.e., to read the whole degrees and the 5′ divisions.

Direction instruments are used chiefly on long lines and in connection with heliotropes or electric lights. The cross-hairs usually consist of two vertical hairs, set so as to subtend an angle of about 25″ to 40″, and two horizontal hairs, set much farther apart and used merely to limit the portion of the vertical hairs to be used in pointing.

In order to steady the theodolite on its support it is sometimes held in position by means of cords or wires which pass over the base near the leveling screws and which are attached to heavy spiral springs. This is equivalent to hanging heavy weights on the base.

38. The Micrometer Microscope.

The construction of the micrometer is shown in Fig. 43. The graduated drum attached to the head of the screw is divided into 60 divisions corresponding to seconds of angle. As the screw head is turned the slide carrying the two parallel hairs in the field of view is moved in a direction perpendicular to the

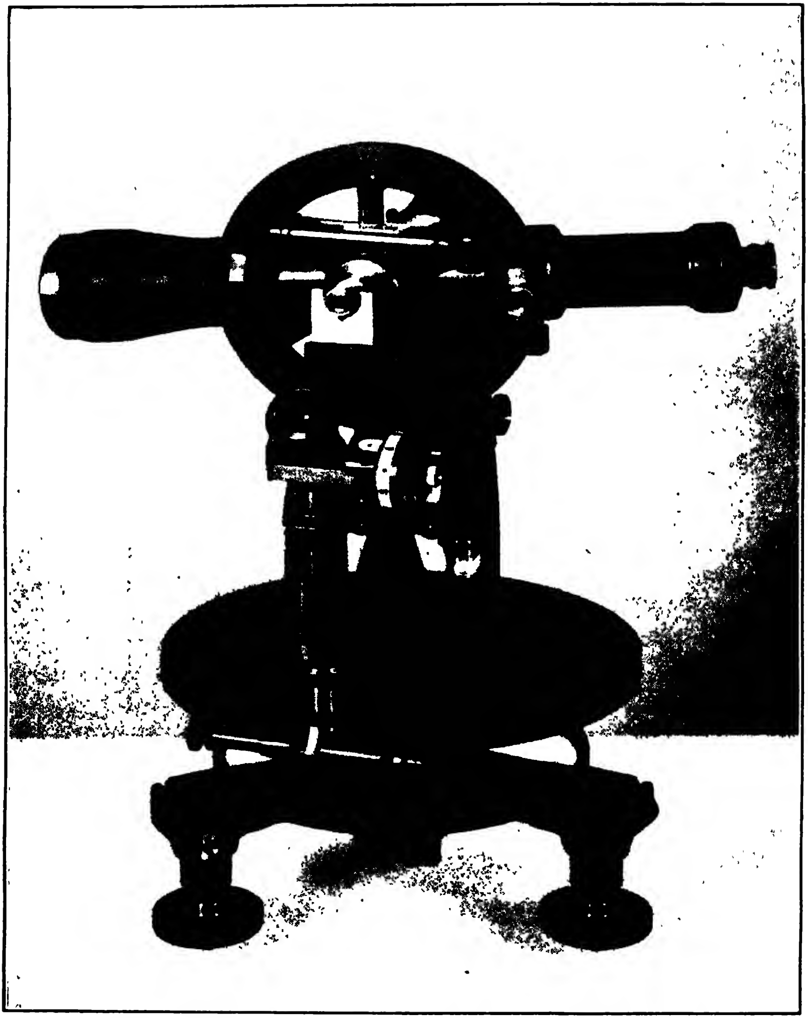

FIG. 42a. First-order Theodolite. (*Coast and Geodetic Survey.*)

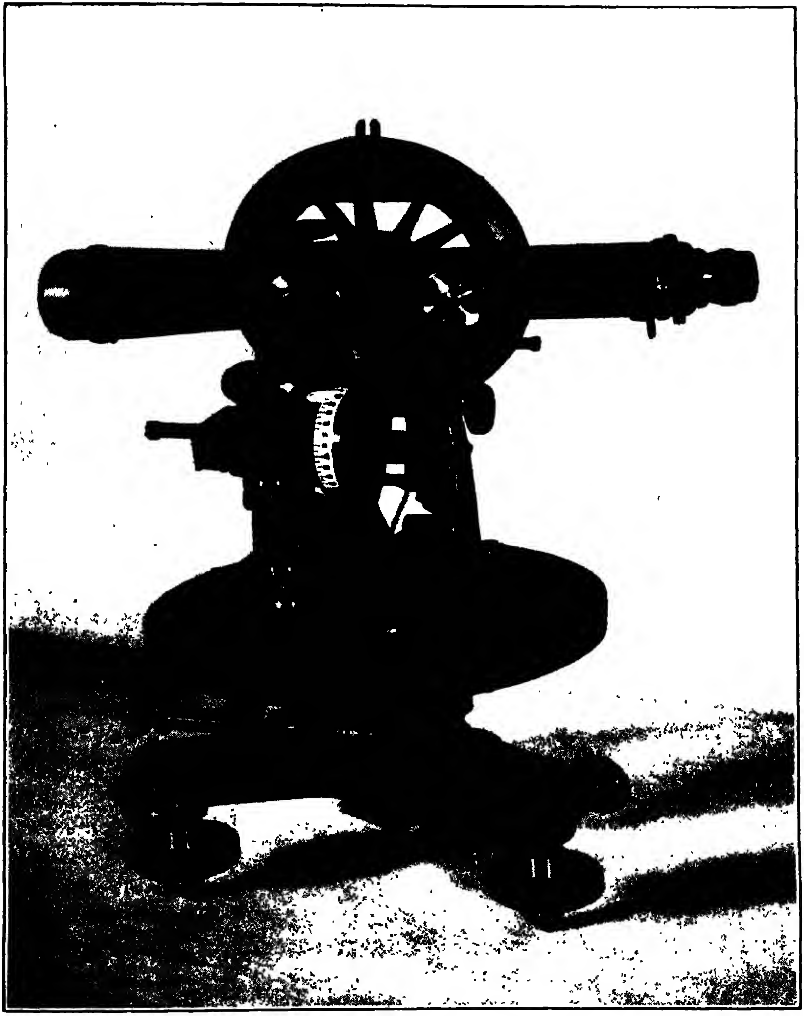

FIG. 42b. Second-order Theodolite. (*Coast and Geodetic Survey.*)

lines of graduation. The hairs are spaced just far enough apart so that a graduation does not quite fill this space but leaves a small white strip on each side of the line when it is centered accurately between the hairs. The pitch of the screw and the focal length of the microscope are so related that one turn of the screw corresponds to one minute. (In the Hildebrand theodolite it is two minutes.) Thus 5 whole turns (or $2\frac{1}{2}$ turns) will carry the hairs from one graduation to the next. The number of minutes or turns passed over may be counted on a

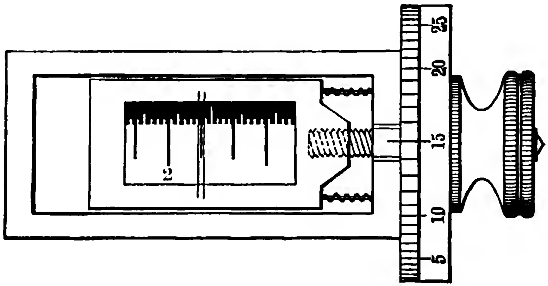

FIG. 43. Micrometer Microscope.

notched (comb) scale placed in the upper portion of the field of view of the microscope. The fraction of a 5′ space which is to be measured is that lying between the zero point of the microscope (indicated by a deep notch in the comb scale and a zero reading of the drum) and the graduated line lying to the apparent left of the zero point. So if we start with the micrometer at zero and turn until the preceding graduation is between the hairs the reading of the comb scale and drum will give this angle directly. It is not necessary that the hairs should actually be set at the zero position before passing to the graduated line; the same reading would be obtained regardless of the position of the hairs before making the setting. It may be easier, however, to understand the operation if it is described as though the hairs really started from the zero position. It should be re-

membered that the microscope inverts the image of the graduations, so that the readings appear to increase from left to right in the field of view of the microscope. The readings on the

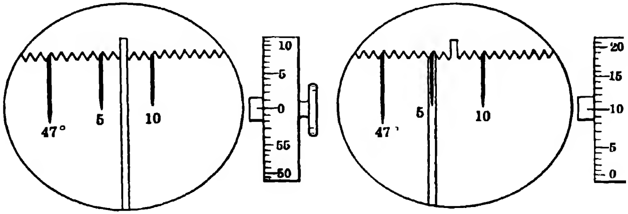

FIG. 44. Showing Micrometer Lines set (*a*) at zero, and (*b*) at the forward reading 2′ 10″

screw head increase as the screw is turned left-handed, and the hairs move back toward the graduation last passed over.

To measure the space between the zero of the microscope and the last line passed over, it is only necessary to turn the screw until the graduation in question bisects the space between the hairs, and then to read the comb scale and the drum. This reading is to be added to the number of the graduation to obtain the direction as shown by this microscope. For example, if the screw is turned two revolutions and 10 divisions in order to center the 47° 05′ line between the hairs (Fig. 44), the reading of the microscope is 47° 05′ + 2′ 10″ = 47° 07′ 10″.

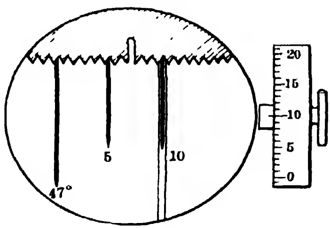

FIG. 45. Showing Micrometer Lines set at back reading 2′ 10″.

Since this gives a direct measure of the direction of the line of sight it is called the "forward" reading.

It should be noticed that since the screw is supposed to make a whole number of turns in passing from one line to the next, the reading on the drum should be exactly the same if the hairs

were set on the 10′ line. If the micrometer screw shown in Fig. 44 is turned 5 times in the right-handed direction, the micrometer hairs will be in the position shown in Fig. 45. This is called the "back reading."

39. The Run of the Micrometer.

If the microscope is perfectly adjusted with respect to the graduated circle, and if the latter is perfectly plane, then 5 whole turns of the screw should carry the hairs from one line to the next and the reading of the screw should be the same on all lines. Since this condition is rarely fulfilled, there is ordinarily a small difference between the forward and the back readings, called the *error of run* of the micrometer.

The forward reading F is taken when the threads are set on the graduation last passed over (25′ in Fig. 46). The back

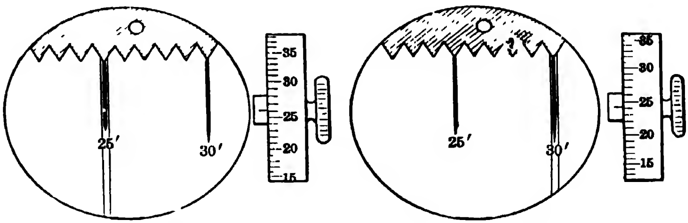

FIG. 46. Forward Reading 1′ 26″.2. FIG. 47. Back Reading 1′ 24″.2.

reading B is taken when the threads are set on the next following graduation. (30′ in Fig. 47.) Suppose the forward reading to be 1′ 26″.2 (on the 201° 25′ line); the direction as obtained from the forward reading would then be 201° 26′ 26″.2. If the screw is now turned to the right so as to carry the hairs to the position shown in Fig. 47, the readings on the screw drum decrease. Suppose that the back reading is 1′ 24″.2. This means that the screw has actually made more than 5 turns, that is, it has passed the reading 26″.2 and gone 2 divisions farther. The direction as derived from the back reading would be 201° 26′ 24″.2. The

true reading, as obtained from a perfectly adjusted microscope, is somewhere between these two results.

Without assuming anything with regard to the actual value of 1 turn of the micrometer screw or 1 division of the drum, the true value may be computed by dividing the angular space between divisions (assuming the graduation to be perfect) by the number of divisions recorded in passing from one graduation to the next. If D is the true angular value of one division, then

$$D = \frac{300''}{300 + F - B} = \frac{300''}{300 + r}$$

where r is the run of the micrometer in seconds divisions, as indicated by the difference between the forward and back readings; it is + if F is greater than B. The true reading is found by multiplying the number of divisions in F by the true angular value of one division, that is,

$$\text{True reading} = F \times \frac{300''}{300 + r}.$$

For the set of readings shown in Figs. 46 and 47 the true reading is

$$86.2 \times \frac{300}{300 + 2} = 85''.629$$
$$= 1'\ 25''.629$$

If it is desired to compute a *correction* to the forward reading F, we have

$$\text{True reading} = F\left(\frac{300}{300 + r}\right) = F\left(\frac{1}{1 + \frac{r}{300}}\right)$$
$$= F\left(1 - \frac{r}{300} \cdots\right) \text{(approx.)}$$
$$F - F\frac{r}{300} \qquad (a)$$

or. correction to $F = -F\dfrac{r}{300}.$

In the preceding example this gives

$$\text{True reading} = 86.2 - 86.2 \times \frac{2}{300} = 86.2 - 0.''575$$
$$= 85''.625+$$
$$= 1'\ 25''.625+$$

Another formula which is sometimes used is derived as follows: if the back reading is also corrected for run the result is

$$300'' - (300 - B)\left(\frac{300}{300 + r}\right)$$
$$= 300'' - \left(300 - B - r + \frac{Br}{300} + \cdots\right)$$
$$= B + r - \frac{Br}{300}. \qquad (b)$$

The mean of (*a*) and (*b*) is

$$\frac{F + B}{2} + \frac{r}{2} - \frac{F + B}{2} \times \frac{r}{300}.$$

If $\frac{F + B}{2} = m$, then the correction to m, the mean, is

$$\frac{r}{2} - m\frac{r}{300}.$$

In the example on page 77 this gives

$$1.0 - 85.2 \times \frac{2}{300}$$
$$1.0 - .568 = .432.$$

Corrected reading $= 1'\ 25''.2 + .432 = 1'\ 25''.632$.

It should be observed that when m is less than $150''$ (the middle of the $5'$ space) the correction is positive (for a positive value of r); when m is more than $150''$ the correction is negative. For any two readings which are equidistant from the middle point of the space the two corrections cancel each other. By a proper distribution of the settings in a complete set of observations it is possible to make the net correction for this microscope equal to zero. This makes the mean for all the microscopes

nearly zero and makes it unnecessary to calculate corrections to each microscope reading. This is done in the program of observations given on page 91.

The discrepancy between the two readings (forward and back) is not only due to run, but also in part to errors in setting the hairs and in reading the drum scale. Sometimes the readings are treated as though these latter errors absorbed the effect of run, and the mean of F and B is taken as the final reading, the effect of run being disregarded entirely.

40. Design of Centers.

The design of the centers of a theodolite has much to do with the accuracy of the final results. In most instruments the

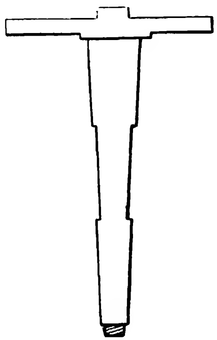

FIG. 48. Center having Single Conical Bearing.

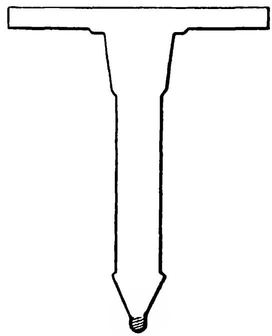

FIG. 49. Center having Two Conical Bearings.

centers have a conical bearing like that shown in Fig. 48, with a small thrust bearing at the bottom. With such an instrument, provision must be made for raising or lowering the center to allow for extreme temperature changes; otherwise, the center will sometimes be too tight or too loose. A center designed in the Instrument Division of the U. S. Coast & Geodetic Survey*

* Designed by Mr. D. L. Parkhurst. See *Jour. Franklin Inst.*, Vol. 206, No. 5. Nov., 1928, p. 623.

has a two-cone bearing, the apexes being coincident. (Fig. 49.) This bearing tends to wear itself in. If there is more friction on the upper cone than on the lower one, it tends to wear until the pressure on the lower cone is increased. If the upper bearing is too loose, the lower cone wears until the center lowers and increases the pressure on the upper bearing. Other features of this theodolite are:

1. A nine-inch horizontal circle (not beveled) read by vertical microscopes, provided with reflecting (angle) prisms for convenience in reading. This makes the instrument more compact. 2. An internal focusing system. A negative lens between the objective and the ocular can be moved so as to bring the image into the plane of the cross-hairs. Therefore, the telescope is short and is tight and dustproof. 3. There is no clamp on the horizontal circle. 4. The center is of steel, turning in a cast-iron socket of the same coefficient of expansion. 5. An improved form of tangent screw. 6. The drums of the micrometers are of frosted glass with a small electric bulb inside for night reading. 7. A new design of vertical circle which is dustproof. 8. The runner of the micrometer is mounted on ball bearings and is practically frictionless. 9. The cross wires are of fine glass fibres; these reflect the light and appear as dark lines in the field of view.

41. Adjustments of the Theodolite.

The adjustment of the levels attached to the alidade is made by means of reversals about the vertical axis of the instrument, exactly as with the engineer's transit.

The adjustment of the stride level is tested by placing it on the horizontal axis, reading both ends of the bubble, and then reversing the level and reading again. The adjusting screws of the stride level should be turned so that the bubble moves halfway back from the second position to the first. When the stride level is so adjusted that it reads the same in either position, it is in correct adjustment, and the horizontal rotation axis may then be leveled by moving the adjustable end of the axis until

the bubble is in the center of its tube. Of course the two adjustments may be made simultaneously. If desired, the stride level may be used also to make the vertical axis truly vertical instead of using the plate levels.

The adjustment of the line of sight in a plane perpendicular to the horizontal axis may be made by reversals about the horizontal axis as in testing an engineer's transit; or it may be made by sighting an object, lifting the telescope out of its bearings, and, after reversing the axis, replacing it in the bearings. If the object first sighted is no longer in the line of sight, the reticle is brought halfway back from the second position toward the first by means of the adjusting screws.

The test of the adjustment of the microscopes is made by measuring the run of each micrometer, taking first a forward reading and then a back reading. This should be repeated several times on the same pair of graduations so as to eliminate accidental errors of setting and reading. The test should also be made in several equidistant parts of the circle. If the run is more than 3″, the microscopes should be adjusted. The short tube carrying the objective of the microscope can be moved in or out of the main tube and the entire microscope can be raised or lowered in its supports. If the image of a circle space is too large, that is, greater than 5 turns of the screw, the objective should be moved away from the circle so as to decrease the angle at the optical center subtended by the two graduations. This is done by pushing the objective tube into the microscope. Doing this causes the image of the graduations to fall in a plane below the micrometer lines. In order to

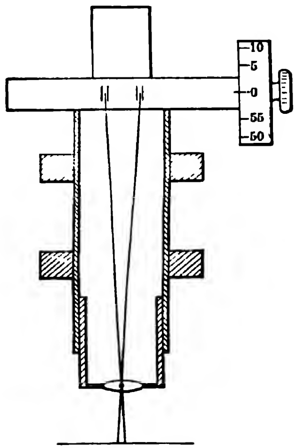

Fig. 50.

bring this image again into the plane of the micrometer lines, the whole microscope must be lowered. The result of this second adjustment is to decrease slightly the effect of the first adjustment. It is advisable, therefore, to overcorrect a little when pushing the objective tube into the microscope tube. It will require several repetitions of this operation to perfect the adjustment for the run of the micrometer.

The adjustment of the drum so that it will read zero when the hairs are in the zero notch may be made by holding the drum firmly with one hand so that it reads zero and turning the screw with the other hand until the hairs are centered in the zero notch. The drum is ordinarily held merely by friction so that this adjustment is easily made. If there is a set screw holding the drum it must be loosened first.

To adjust the B microscope so that it is exactly 180° from the A microscope set the A micrometer to read zero and then move the alidade until a graduation is between the hairs of the microscope. Then set the hairs of microscope B on the opposite graduation and, holding the screw in this position, turn the drum until it reads zero. If the position of the comb scale is out of adjustment it will be necessary to correct this first. This adjustment can usually be made by means of a screw in the end of the micrometer box which shifts laterally the frame carrying the comb scale.

42. Effect of Errors of Adjustment on Horizontal Angles.

The effect of errors due to the inclination of the horizontal axis to the horizon, and those due to the imperfect adjustment for collimation (line of sight), are not independent of each other. These errors are usually so small, however, that it is permissible to compute their effect separately, as though only one existed at one time. In Fig. 51, Z is the true zenith and Z' the point where the vertical axis of the instrument prolonged pierces the celestial sphere. S is a point whose altitude is h. Assuming that the horizontal axis makes an angle i with the horizon, and that all other errors are zero, then from the figure it will be seen that we

may write

$$\frac{\sin Z'}{\sin i} = \frac{\sin HS}{\sin Z'S},$$

or, with sufficient accuracy,

$$Z = i \tan h, \qquad [11]$$

where h is the angular altitude of the point sighted.

It appears, then, that for each point sighted there should be a correction to the circle reading equal to $i \tan h$. Triangulation points are usually so nearly on the horizon, and by careful attention to the leveling the error i may easily be kept so small, that

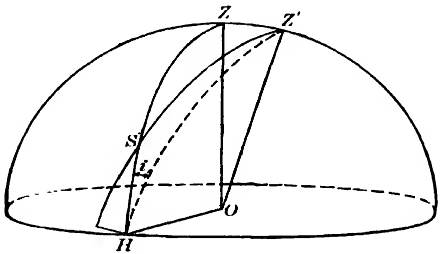

FIG 51. Error in Horizontal Axis.

there is seldom any necessity for applying the correction for inclination except for such observations as those on a circumpolar star for azimuth.

In the preceding paragraph it is assumed that the vertical axis is truly vertical, the graduated circle being horizontal, while the horizontal axis is not horizontal. If the two axes are at right angles to each other, but the vertical axis is inclined to the true vertical by a small angle i, owing to imperfect leveling, it may be shown, by a diagram similar to Fig. 51, that the same correction applies to this case also.

The error of a horizontal direction due to an error of collimation may be computed as follows: Let the error in the sight line be represented by c; then, when the axis of collimation (Fig. 52)

traces out the great circle ZN, the line of sight traces out the parallel circle SA, which is c seconds from ZN. If S be any point toward which the cross-hair is pointing, and if arc SN be drawn perpendicular to ZN, then the error in direction, or the angle at Z, is found from the equation

$$\frac{\sin Z}{\sin N} = \frac{\sin c}{\sin ZS}$$

or, since $\angle N = 90°$, $$Z = c \sec h. \qquad [12]$$

Each direction should therefore be corrected by the quantity

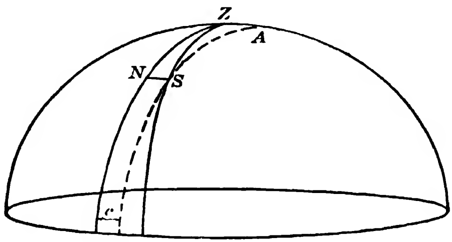

FIG. 52. Collimation Errors.

$c \sec h$. On account of the small value of c in a well-adjusted instrument this correction is necessarily small; furthermore, it is usually eliminated from the final result by the method employed in making the observations.

43. Errors of Graduation.

The errors in the positions of the graduations on the circle may be either periodic or accidental. In a good instrument these errors should not be more than about 1″, or in any case 2″. Since the total angular space about the center is a fixed amount, positive periodic errors in one part of the circle must be balanced by negative periodic errors in another part of the circle. If, when measuring directions or angles, the circle is shifted between sets of observations so that the readings on any signal are distributed uniformly around the circle, the errors must

balance to a large extent; the greater the number of such readings the more nearly will the final mean be free from errors of graduation. The accidental errors follow no law except that of the theory of probability. Their effect on the final angle is diminished as the number of readings is increased, that is, as the number of lines used is increased.

44. Errors of Eccentricity of Circles.

Errors of this kind are due to the fact that the center of the alidade does not coincide with the center of the graduated circle. The effect of this error on the readings is periodic as will be seen

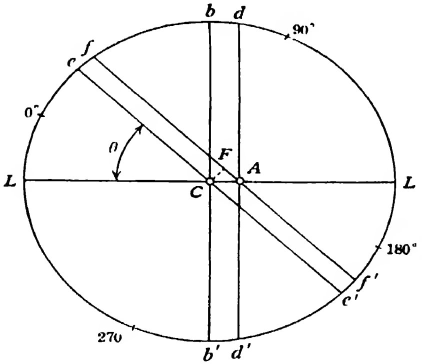

FIG. 53. Showing Errors due to Eccentricity of Circles.

by Fig 53. If C is the center of the graduated circle and A is the center about which the alidade revolves, then the distance CA is interpreted as an arc bd in seconds on the graduated scale and is called the eccentricity. LL' is the line of centers. If a vernier or a microscope is at L it reads the same as it would if there were no error of eccentricity; whereas at b it is in error by $e = bd$, the full amount of the eccentricity, CA. In intermediate positions the error of reading will be found to equal $e \sin \theta$, θ being the angle LCe.

A microscope supposedly at b will actually be at d and will

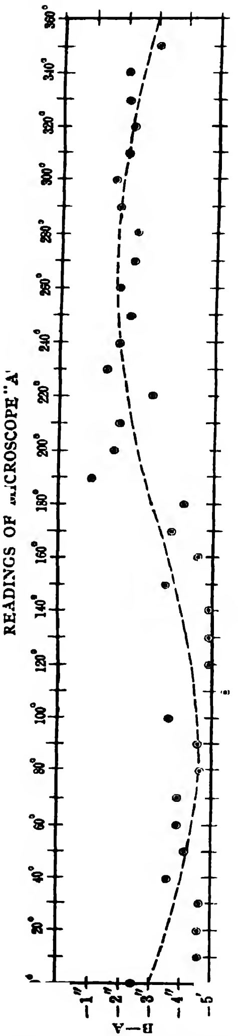

FIG. 54. Test of a Theodolite.

read too much by the amount $+e$. The opposite microscope will be at d' instead of b' and will read too small by $-e$. The mean of the two readings is free from this error e. A microscope at e reads an amount corresponding to f, which is too large by the amount $+e \sin \theta$; the opposite microscope reads f', which is too small by the amount $-e \sin \theta$. The mean of the two microscopes is therefore free from errors of eccentricity. That is, if we take the mean of the readings of opposite microscopes, the result is the same as though there were no eccentricity of the circle.

It may be shown that the mean of three microscopes 120° apart, or any number of microscopes equally spaced around the circle is free from the effects of eccentricity.

In general each microscope reading requires the correction

$$+e'' \sin (z + E)$$

where z is the microscope reading and E is the angle between 0° and the line of centers.

44a. Test of Theodolite.

When a new theodolite is being used it is desirable to test the accuracy of its graduated circle and the microscopes. This may be conveniently done by taking readings around the circle at equal intervals as follows: Set microscope A at 0° 00′ 00″ and read microscope B. This will differ from the A reading by the constant angle (α) between the microscopes, the errors of graduation, eccentricity and run of mi-

crometer. Then set microscope A on 10° and read B; and so on until the entire circle has been covered. The differences, $B - A$, may be plotted on coördinate paper and the resulting curve studied to judge the amount of the different errors. It is possible to calculate these, but the main thing desired is to know in a general way whether the instrument is of good or poor quality, so it is doubtful whether it is worth while to calculate all of the component errors.

If the algebraic mean is taken of all the $B - A$ readings the result is the constant angle (α) between microscope B and a point 180° from microscope A, since the eccentricity terms will cancel each other. If this amount is subtracted from the values of $B-A$ the remainder in each case is composed of errors of eccentricity, graduation and run. These may be separated by computation. If the plot shows a distinct sine curve, it indicates eccentricity combined perhaps with periodic errors of graduation. In a good instrument these errors should be small. Variations of points from the mean curve shows accidental errors of graduation and micrometer errors. If the points show variations from the mean curve of more than $1''$ or $2''$, the graduation of the circle is probably of inferior accuracy. Figure 54 shows the results of a test of a theodolite. The B microscope is about $3''.3$ less than 180° from microscope A. A smooth curve through the points shows that there is a small amount of eccentricity, probably combined with graduation errors. The variations from the mean are small except in one instance, where there appears to be an accidental error of more than $1''$.

An approximate value for the amount of eccentricity and the approximate position of the line of centers may be computed from four equidistant readings,* say 0°, 90°, 180° and 270°. If we put $B - A = n$ then each pair of readings gives an equation of the form

$$n = \alpha + 2\,e \sin (z + E).$$

* Chauvenet, Spherical and Practical Astronomy, Vol. II, p. 40.

Let the four values of n be

$$\begin{aligned}
n_0 &= \alpha + 2\,e \sin E &&= \alpha + 2\,e \sin E \\
n_1 &= \alpha + 2\,e \sin (E + 90^\circ) &&= \alpha + 2\,e \cos E \\
n_2 &= \alpha + 2\,e \sin (E + 180^\circ) &&= \alpha - 2\,e \sin E \\
n_3 &= \alpha + 2\,e \sin (E + 270^\circ) &&= \alpha - 2\,e \cos E
\end{aligned}$$

Therefore $$n_0 - n_2 = 4\,e \sin E$$

and $$n_1 - n_3 = 4\,e \cos E$$

from which we may compute E and e. The value of α is the mean of the four values of n.

Take, for example, the following four readings,

A	B	$B - A$
0°	$179^\circ\ 59'\ 57''.5$	$n_0 = -2''.5$
90°	$269\ \ 59\ \ 55\ .4$	$n_1 = -4\ .6$
180°	$359\ \ 59\ \ 55\ .8$	$n_2 = -4\ .2$
270°	$89\ \ 59\ \ 57\ .4$	$n_3 = -2\ .6$

From which we obtain

$4\,e \sin E = +1''.7$	log 0.2304
$4\,e \cos E = -2''.0$	log 0.3010 n
$E = 139^\circ\ 38'$	log tan E 9.9294 n
$e = 0''.66$	log cos E 9.8819
	log $4\,e$ 0.4191
	log 4 0.6021
	log e 9.8170

Therefore each microscope requires the correction for eccentricity

$$+0''.7 \sin (z + 139^\circ\ 38').$$

The value of α is $-3''.5$.

45. Method of Measuring Angles. — Repeating Instrument. In measuring the angles of a triangulation with the repeating instrument, it is common practice to measure the angle six times,

beginning with the left-hand signal of a pair and measuring toward the right; then, after reversing the telescope, to measure six times beginning with the right-hand signal and ending on the left-hand signal. In order to eliminate any possible error due to drag of the lower plate by the alidade or the center, the telescope is always turned in a clockwise direction. In the second half of the set of 12 angles the telescope is moved from the right-hand signal toward the right through an angle of $360° - x$ until it sights the left-hand signal. This brings the vernier almost exactly to the same reading that would be obtained by turning toward the left, but it differs in the mechanical action.

The reversal of the telescope is for the purpose of eliminating errors due to non-adjustment of the line of sight and of the horizontal axis. It should be observed that it does not eliminate the error due to imperfect leveling of the circle. It is important to watch the levels and to keep them adjusted and centered. They may be re-leveled without affecting the angles whenever the lower clamp is loose. The change in direction of the measurement of the two half-sets (left to right and right to left) is intended to eliminate twist of the supporting tower. This is very small with the modern towers, but any twist which does occur and which takes place at a uniform rate is eliminated by this method of observing.

Errors due to faulty graduation may be eliminated to a large extent by changing the initial reading of vernier A in each set in such a way as to distribute the readings uniformly around the entire circle. If the initial reading is advanced each time by an amount equal to $\frac{360°}{mn}$ (where m is the number of sets to be measured, and n is the number of verniers) the readings will be distributed uniformly about the circle. For example, if 6 sets are to be taken with a 2-vernier instrument the reading of vernier A should be advanced 30° each time. Errors in the graduation of the verniers can be partially eliminated by so arranging the

initial settings that the readings are distributed uniformly over the smallest space on the graduations. For example, if 6 sets

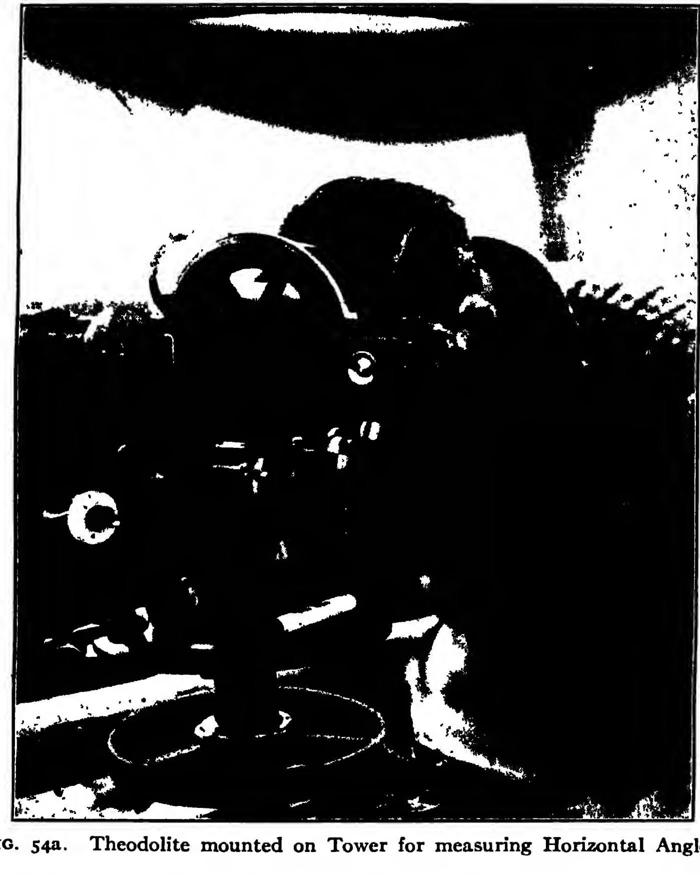

Fig. 54a. Theodolite mounted on Tower for measuring Horizontal Angles.

are to be measured and the smallest division is 10′, the initial settings of vernier *A* would be 0° 00′ 00″, 30° 01′ 40″, 60° 03′ 20″, etc.

Direction Instrument.

When the direction theodolite* is used, the telescope is first sighted approximately at one of the signals selected as the initial for the series, and the circle is then turned so as to bring any desired reading under the index microscope. The pointing on the initial signal is then perfected and all the microscopes are read. The telescope is then pointed on the second signal, (to the right), the circle remaining in the same position, and the microscopes are again read. This process is continued until the last signal has been sighted and the microscopes read. The telescope is then inverted, the last signal again sighted, and the readings taken. The signals are sighted in order, from right to left, until the initial point is again sighted. This completes the observations in the first "position." The number of positions to be used depends upon the accuracy desired. For first-order triangulation 16 positions are required.

Just as with the repeating theodolite, the reversal of the telescope of the direction instrument eliminates errors due to faulty adjustment of the line of sight and the horizontal axis. The instrument must be kept accurately leveled as this error is not eliminated by reversal. The change in direction from left to right and then from right to left eliminates twist. The drag is eliminated by bringing the telescope up to the initial point from a position a little to the left, so that the drag is taken out before the series is begun; on the second half the telescope is turned to the last signal in the series from a position a little to the right of it. The change in the reading on the initial signal and the distribution of the micrometer readings over the smallest space of the circle are made in a manner similar to that explained in the preceding article. In the following tables will be found the initial settings for a 3-micrometer and a 2-micrometer theodolite, for 16 positions and 8 positions respectively. When

* For the methods used by the U. S. Coast & Geodetic Survey see Special Publication No. 120, by Major C. V. Hodgson.

THEODOLITE SETTINGS WHEN 16 POSITIONS OF THE CIRCLE ARE USED

With a 3-micrometer theodolite.		With a 2-micrometer theodolite.	
Position No.	Setting.	Position No.	Setting.
	° ′ ″		° ′ ″
1	0 00 40	1	0 00 40
2	15 01 50	3	11 01 50
3	30 03 10	3	22 03 10
4	45 04 20	4	33 04 20
5	64 00 40	5	45 00 40
6	79 01 50	6	56 01 50
7	94 03 10	7	67 03 10
8	109 04 20	8	78 04 20
9	128 00 40	9	90 00 40
10	143 01 50	10	101 01 50
11	158 03 10	11	112 03 10
12	173 04 20	12	123 04 20
13	192 00 40	13	135 00 40
14	207 01 50	14	146 01 50
15	222 03 10	15	157 03 10
16	237 04 20	16	168 04 20

THEODOLITE SETTINGS WHEN 8 POSITIONS OF THE CIRCLE ARE USED

With a 3-micrometer theodolite		With a 2-micrometer theodolite.	
Position No.	Setting.	Position. No.	Setting.
	° ′ ″		° ′ ″
1	0 00 40	1	0 00 40
2	15 01 50	2	22 01 50
3	30 03 10	3	45 03 10
4	45 04 20	4	67 04 20
5	52 00 40	5	90 00 40
6	67 01 50	6	112 01 50
7	82 03 10	7	135 03 10
8	97 04 20	8	157 04 20

the above programs are carried out it is found to be unnecessary to apply corrections for run. (See page 78.)

When making bisections of the signal or when making micrometer settings, it is advisable to proceed as rapidly as possible without making blunders. The observer should make the set-

FIG. 54b. Triangulation with Direction Instrument.

ting at once, should decide promptly whether it is correct, and then proceed to the next operation. If much time is spent in watching signals to see if they remain bisected and in resetting micrometers to see if the same reading is obtained several times in succession, the result is liable to be less accurate rather than more accurate. The reason for this is that the instrument and its support are continually in motion owing to temperature changes and other causes. The longer the interval between pointings the greater these errors are liable to be. The errors entering the result from these causes are probably larger than

those which are being reduced or eliminated by the perfection of settings and pointings. The checks of the microscope readings are therefore likely to be deceptive.

Rejecting Observations. The limit for rejecting a single observed direction (U.S.C. & G.S.) is 4″ from the mean. This value is found from experience to be a safe one to use and is much more readily applied than any of the usual rules based on the theory of probability.

Errors in Azimuth.

Investigations of the accumulated error in the azimuth of a chain of triangles shows that there is a systematic tendency to twist in azimuth; this is due to the unequal heating of different parts of the theodolite during observations made in daylight. On arcs extending north and south the error is greater on the eastern side of the chain than on the western side. This is apparently due to the fact that the observations were formerly made almost wholly in the afternoon, and the instrument on the eastern side of the triangulation cannot be so completely shielded from the sun as it can be when on the western side of the chain. Observations made at night show no such systematic difference between the two sides of the triangulation.

The error in azimuth just mentioned can be much reduced by turning the graduated circle 180° between "positions." This is provided for in the program of observations already described. At the end of each series of readings the telescope is in the opposite position from what it was at the beginning. If it is left in this position (instead of turning it back to the normal position each time before beginning a new series) it will then be necessary to turn the circle nearly 180° in order to bring the next initial reading in the list under the index microscope.

For a method of correcting azimuths for the error of accumulated twist, see page 259.

In this connection it is important to notice a source of error in azimuth which becomes systematic under some circumstances. Whenever a triangulation line passes close to the side

of a hill or mountain there is a bend in the ray of light due to horizontal refraction. This is probably because the variation in density of the layers of air has a tendency to follow the perpendicular to the surface rather than the true vertical. This error of course affects the closure of the triangles containing such lines. If a survey is carried along a river the same kind of error is likely to occur many times over and thus introduces error of a systematic character into the final results. (See Bull. No. 56, National Research Council; J. L. Rannie, Geodetic Survey of Canada.)

Time for Measuring Horizontal Angles.

It was formerly the practice to measure angles only during the day time, and only during that part of the day when the signals appear steady, that is, during the latter part of the afternoon. On the triangulation of the 98th meridian, observers were instructed to measure angles whenever the results indicated that the required accuracy was being obtained, and not to rely upon the appearance of the signals. The results showed that good observations can often be made when the appearance of the signals would seem to indicate that good results could not be obtained. Triangulation at night was also tried in comparison with the daylight observations. The results were almost uniformly more accurate than those of day observations; they were also obtained more economically because there were fewer delays. Observations on heliotropes are often held up for hours and days at a time when the sun is obscured by clouds, whereas electric lights can be seen under nearly all conditions except in foggy weather. The greater part of the first-order triangulation at the present time is done at night. This has the additional advantage that the observations for astronomical azimuths can be combined with the measurement of horizontal angles, which results in a considerable saving.

Forms of Record.

The following are forms of record which may be used for horizontal angles in a triangulation.

HORIZONTAL ANGLES. DIRECTION INSTRUMENT

Station, Corey Hill. Date, May 21, 1907. Observer, A. N. Recorder, W. R. N. Instrument No. 31. Position No. 2.

Station observed.	Time.	Tele-scope. D. or R.	Micro.	Circle. ° ′	Circle. F.	Circle. B.	Run.	Mean.	Cor. for run	Cor'd secs.
	h m									
Blue Hill	4 30	Dir.	*A*	15 01	51 5	50 5				
			B		54 0	53 7				
			C		49 0	48 5				
					51.5	50.9	0 6	51 2		
Prospect		Dir.	*A*	138 30	20 9	20 5				
			B		22 0	21 5				
			C		18 1	18 0				
					20 3	20.0	0 3	20 2		

HORIZONTAL ANGLES. REPEATING INSTRUMENT.

Station, Corey Hill. Date, May 21, 1907. Observer, J. N. B. Instr. B. & B., No. 1567.

Station.	Time.	Tel.	Rep.	Ver. A.	B.	Mean.	Angle.	Mean.
	h m			° ′ ″	″	″	° ′ ″	° ′ ″
Blue Hill to Prospect	3 20	D	0	0 00 00	00	00		
	P.M.		1	123 28 10	20	15		
			6	*20 49 40	40	40	123 28 16.7	
		R	0	20 49 40	40	40		
			6	0 00 10	10	10	123 28 15 0	123 28 15.8

*** *Note.* — Since the angle is over 120 degrees the *A* vernier has passed 360 degrees twice in the six repetitions. In computing the mean we divide the 720 degrees by 6 mentally and write down 12 —, then divide the 20 degrees by 6, add the whole degrees to 120, and then divide the minutes and seconds. Observe that when six repetitions are used, the remainder, when dividing the degrees by 6, gives the first figure of the minutes, i.e., 20 degrees ÷ 6 = 3 degrees in the mean, plus 2 degrees to be carried to the minutes column giving 20 minutes. Similarly in dividing the minutes by 6 the remainder is the tens place in the seconds.**

46. Accuracy Required.

The accuracy required in the different "orders" of triangulation has been stated (page 4). After the triangulation has been adjusted by least squares, the "probable error of an observed direction" becomes known. This is in some respects a better measure of the precision than the triangle closure, which was used in the field to test the accuracy of the results. The following brief list, taken from a more extended list in Special Publication No. 19, U. S. Coast & Geodetic Survey, will show the degree of accuracy actually reached on several different triangulation arcs.

Section.	Probable error of an observed direction	Average closing error of a triangle.	Max. cor. to direction	Maximum closing error of a triangle.
	"	"	"	"
Nevada — California .	±0 23	0 57	0 60	1 57
New England	±0 26	0 75	1 17	2 02
Eastern Oblique Arc.	±0 30	0 78	0 74	2 73
Holton Base net	±0 34	0 79	0 84	2.28
Atlanta base to Dauphin Island-base. . . .	±0 36	1 10	0 84	2.69
Lampasa base to Seguin base	±0 45	1 13	1 96	3 31
Calif. — Washington Arc .	±0 53	1 22	2.03	6 35

47. Reduction to Center.

If some of the lines from a station are obstructed, or if the nature of the signal is such that the theodolite cannot be placed directly over the center, it becomes necessary to set the instrument over a point at one side of the center, called an "eccentric station," and to measure the angles from this new point. These angles are to be measured with the same degree of precision as though they were taken at the center; and, in addition, sufficient measurements must be made to enable the computer to find the true central angles. These additional measurements include the distance from the center to the eccentric station and the angle (at the instrument) between the center and some one of the distant signals.

It is also necessary to know approximately the lengths of the sides of the triangles involved. These can usually be computed with sufficient accuracy from a known base by means of other angles which are not dependent upon eccentric stations.

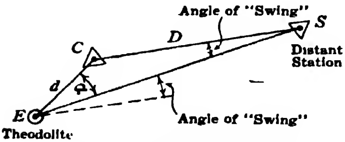

FIG. 55. Eccentric Station.

In Fig. 55, C is the center mark, E is the theodolite at the eccentric station, and S is one of the distant stations. $CE = d$ is the *eccentric distance* and may be measured with a tape. CS is the triangulation line $(=D)$ supposed to be approximately known. Angle $CES = \alpha$ is the *direction* of S referred to EC as an initial direction. It is measured with the theodolite and is always counted toward the right. For any other signal it may be computed by combining the angle CES with the angle between S and the station in question. Angle $ESC = S$ is the angle of "swing" and is the correction to be applied to the direction ES to obtain the direction CS.

In order to compute the angle S solve the triangle CES (law of sines), obtaining

$$\sin S = \frac{CE}{CS} \sin \alpha$$

$$= \frac{d}{D} \cdot \sin \alpha$$

or

$$S'' = \frac{d \sin \alpha}{D \text{ arc } 1''}.$$

From the angles between distant stations and the angle connecting one of these with the initial station (C) we may compute the direction of every signal sighted, referring these directions to the initial station as 0°. For convenience in applying the corrections, we may retain the seconds and fractions, but this is not necessary for the purpose of taking out the log sines.

Then for each distant station we compute the correction (S'') with its appropriate algebraic sign. An examination of the direction (α) and the algebraic sign of the correction (S'') at each station will show that if sin α is given the proper algebraic sign according to whether α is under or over 180° the resulting sign of the correction (S'') will be that which must be added to

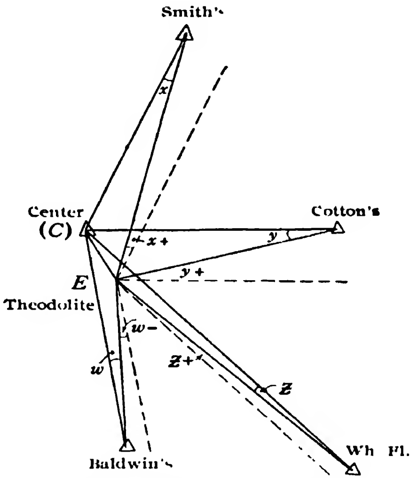

FIG. 56. Reduction to Center.

the observed direction to give the true direction from the center. After the correction (S'') has been added (algebraically) to each direction the results are the true directions from the center. In Fig. 56 these are shown by the dotted lines. If the dotted lines are moved parallel to themselves until E coincides with C then these dotted lines coincide with the true directions from the center. If the true angle between any two stations is desired, it is found by taking the difference between the two corrected directions of the stations in question.

FIG. 56a. Vertical Circle. (*Coast and Geodetic Survey.*)

50. Vertical Collimator.

The vertical collimator is an instrument designed and used for the special purpose of placing a theodolite (or a tower) exactly over the station mark on the ground. The older pattern (Fig. 57) was used on the top of the tower and consists of a tele-

FIG. 57. Vertical Collimator. (*Coast and Geodetic Survey.*)

scope mounted in the vertical position and used to sight directly at the station mark. It is provided with a spirit level and can be turned on its own axis for the purpose of adjustment, similar to the manner in which a Wye level is rotated in its bearings for the adjustment of the cross-hairs. The telescope may be removed from its sleeve and a plunger inserted in its place. The point of the plunger shows the point on the cap of the tripod over which the theodolite should be placed.

The new pattern of vertical collimator (U.S.C. & G.S.) is

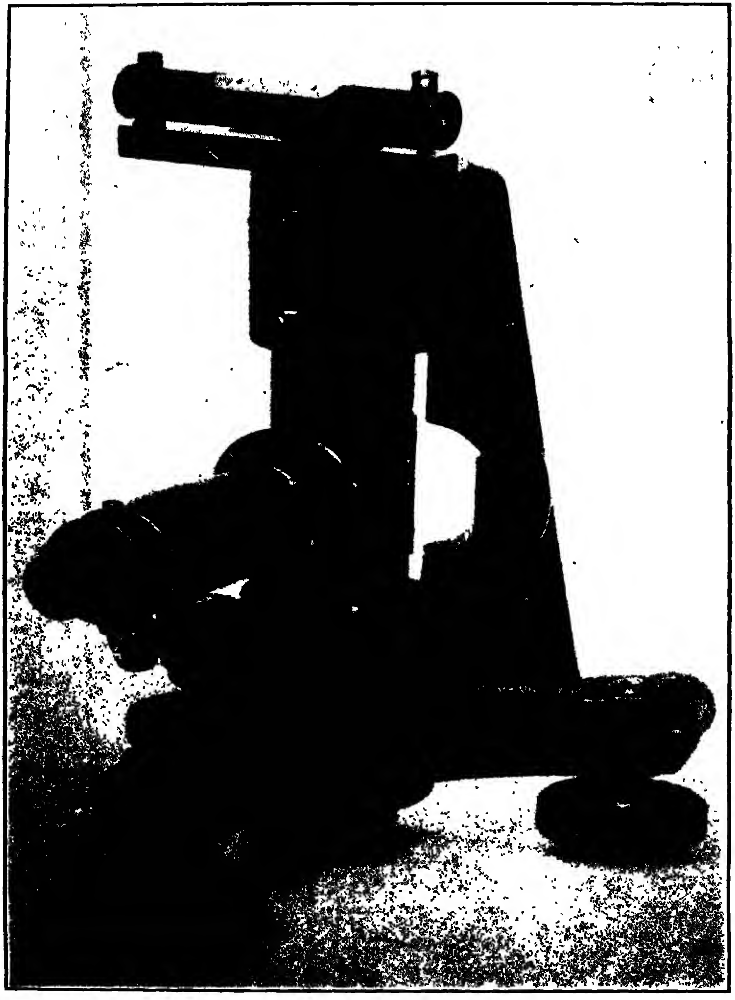

FIG. 58. Vertical Collimator. *(C. L. Berger & Sons.)*

placed on an ordinary tripod on the ground and the telescope sights vertically upward (Fig. 58). A prism between the objective and the eye-piece enables the observer to see the top of the tower by looking horizontally through the eye-piece. When the instrument is adjusted and is placed over the center mark, the theodolite can be placed directly in position so as to cover the intersection of the cross-hairs in the vertical collimator.

PROBLEMS

Problem 1. The circle of an alt-azimuth instrument is graduated into 10-minute spaces. The pitch of the micrometer screw is such that two turns are required to move the hairs from one graduation to the next. The head of the screw is divided into minutes and each minute into 10-second spaces. The forward reading (on the 260° 10′ line) is 4′ 03″; the back reading (on the 260° 20′ line) is 3′ 55″ What is the run of this micrometer? What is the correct reading?

Problem 2. The forward reading in a microscope is 2′ 10″.5, on the 70° line; the back reading is 2′ 12″.0 on the 70° 05′ line. What is the correct reading of the direction?

Problem 3. The readings of a striding level on a theodolite show that the horizontal axis is inclined 1.5 divisions, the left end being higher. What error will this cause in the azimuth reading on the pole star, at an altitude of 41° 20′, if the value of one division of the level is 10″.0?

Problem 4 If a horizontal angle is measured between a mark 12° above the horizon and bearing N 45° W, and the pole star, 41° altitude, what is the error in the angle produced by an error of 8″ to the right in the (collimation) adjustment of the vertical cross-hair.

Problem 5. The angle between stations *A* and *B* is measured from station *E* and found to be 71° 10′ 19″.5. The angle from *O*, to the right, to station *A* is 110° 15′. The distance *OE* is 7.460 meters. *OA* is 17,650 meters and *OB* is 24,814 meters. (Fig. 59.) Reduce the angle to the center *O*.

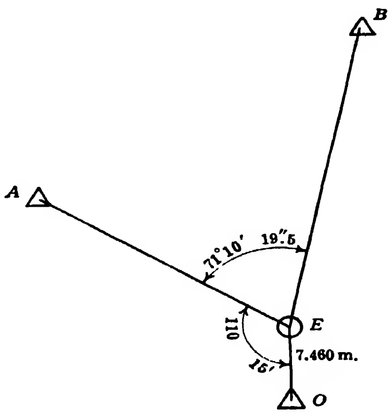

Fig. 59.

Problem 6. The angle from the center (6.00 meters away) to signal *A* (clockwise) is 31° 10′ 29″.0; from *A* to *B* is 61° 59′ 00″.0; from *B* to *C* is 129° 29′ 17″.2. The distances from the center

to A, B, and C, are 8000^m., 9000^m., and 9500^m., respectively. Reduce these angles to center.

Problem 7. If the diameter of a graduated circle is 8 inches, and the center about which the telescope rotates is one ten-thousandth of an inch from the center of the graduated circle what is the maximum error of eccentricity that will affect any single microscope reading?

CHAPTER IV

ASTRONOMICAL OBSERVATIONS

51. Astronomical Observations — Definitions.

In every trigonometric survey, whether made for scientific purposes or for the purpose of making maps, it is essential that some of the triangulation points be located on the earth's surface by means of their astronomical coördinates. In determining the earth's size and figure by measuring arcs on the surface it is essential that the curvature be determined by means of astronomical observations. If the triangulation is used to control the accuracy of a topographical survey, the astronomical work furnishes the data necessary for correctly locating and orienting the map on the earth's surface. The astronomical data also furnish a means of detecting the accumulated twist of a chain of triangulation, and of correcting the azimuth at intervals along the line. Astronomical observations are also frequently made in order to supply data to be used in other measurements, as, for example, when rating chronometers for gravity or magnetic observations. These astronomical observations form a distinct branch of geodetic work.

It will be assumed that the student has a general knowledge of astronomy, and only such definitions will be given as are essential in viewing the subject from the standpoint of the geodesist. The astronomical observations which it is important for us to consider include the determination of the following four coördinates: (1) time, (2) longitude, (3) latitude, and (4) azimuth. Before describing the instruments and methods, we will define the following terms which are to be employed.

The *vertical* at any point on the earth's surface (OZ, Fig. 60) is the direction in which the force of gravity acts at that point.

In general it does not perfectly coincide with the normal to the spheroidal surface, and hence there is a difference between the astronomical coördinates and the geodetic coördinates. The deflection of the plumb line from the normal at any place is called the *station error*. The point vertically overhead (Z) is called the *zenith*. We may consider that the universe is bounded by a

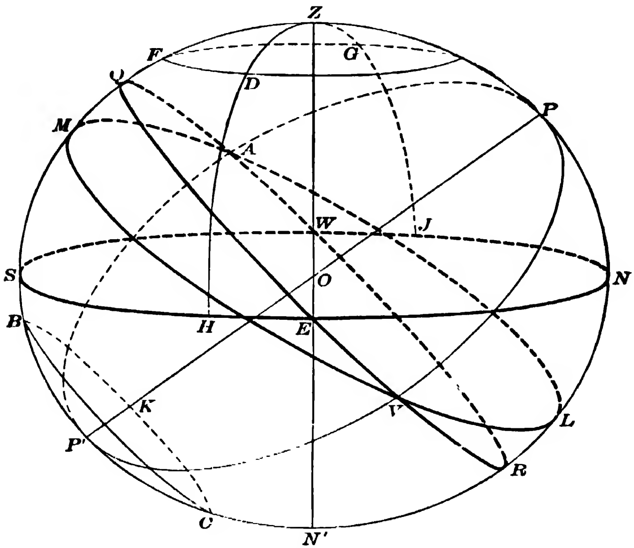

FIG. 60. The Celestial Sphere.

sphere of infinite radius, and that the zenith is the point where the vertical pierces that sphere. The *horizon* ($NEHS$) is the great circle on the celestial sphere which is everywhere 90° from the zenith. Its plane passes through the observer and is perpendicular to the vertical line. Any plane which contains the vertical line cuts from the sphere a *vertical circle* (HDZ).

The earth's rotation axis, prolonged, pierces the sphere in two points, called the *north celestial pole* (P) and the *south celestial*

pole (P'). The great circle which is everywhere 90° from the poles is the *celestial equator* (QVR). Any plane through the axis or parallel to it cuts from the sphere an *hour circle* (PVP'). The vertical circle which passes through the celestial pole is called the *meridian* (SQZ). If the vertical does not intersect the earth's axis, the meridian plane cannot contain the axis but is parallel to it. The *prime vertical* is a vertical circle perpendicular to the meridian. The *ecliptic* is a great circle cut by the plane of the orbital motion of the earth (MVL). That point on the sphere where the ecliptic and the equator intersect, and where the sun passes (in March) from the southern to the northern hemisphere, is called the *vernal equinox*.

The *altitude* (h) of a point is its angular distance above the horizon. Its *zenith distance* (ζ) is the complement of the altitude. The *azimuth* (Z) of a point is the horizontal angle between the meridian and the point. It is usually reckoned from the south point of the horizon, right-handed, from 0° to 360°. The *declination* (δ) of a point is its angular distance north (+) or south (−) of the equator. Its *polar distance* (p) is the complement of the declination. *The hour angle* (t) of a point is the arc of the equator measured from the meridian westward to the hour circle through the point. The *right ascension* (α) is the arc of the equator measured from the vernal equinox eastward to the hour circle through the point.

The astronomical *latitude** (ϕ) of a place is the angular distance of the zenith north or south of the equator, or, in other words, the declination of the zenith. The longitude (λ) of a place is the arc of the equator between the observer's meridian and a primary meridian, as Greenwich or Washington.

52. The Determination of Time.

The determination of time, practically considered, means the determination of the error of a chronometer on the local sidereal time at the station. The sidereal time (S) at any instant is the hour angle of the vernal equinox; it is usually expressed in hours,

* For geodetic latitude see p. 166.

minutes, and seconds. From a consideration of the definitions of sidereal time, hour angle, and right ascension it is evident that the first equals the sum of the other two; that is,

$$S = \alpha + t. \qquad [15]$$

When the star is on the meridian, t is obviously equal to zero, and we have

$$S = \alpha, \qquad [16]$$

that is, the right ascension of any star is equal to the sidereal time at the instant when that star is passing the meridian. If we note the chronometer reading when a certain star is passing the meridian, we know that the local sidereal time (or true chronometer reading) at that instant is the same as the right ascension of that star as given for that date in the Ephemeris,* and that the error of the chronometer is the difference between the two. The determination of time with a transit mounted in the plane of the meridian depends upon the foregoing principle.

53. The Portable Astronomical Transit.

The instrument chiefly used for determining time and longitude in geodetic work is the portable transit. This class of work necessitates carrying the instrument to many stations located in places which are difficult to reach; hence it should be light enough to be easily transported. The small size of the transit, however, does not necessarily imply inferior accuracy in the results; it is found by experience that comparatively small instruments, when properly handled, give results of great accuracy. Indeed, the very fact that the instrument is light is a point in its favor, for this makes it easier to reverse, and obviates certain difficulties encountered in using large instruments in observatories, for example, the error due to flexure, or those due to temporary strains caused by reversal of the instrument. The portable transit is usually mounted on a brick or concrete pier, or on a heavy wooden support.

* The American Ephemeris and Nautical Almanac, published by the Navy Department.

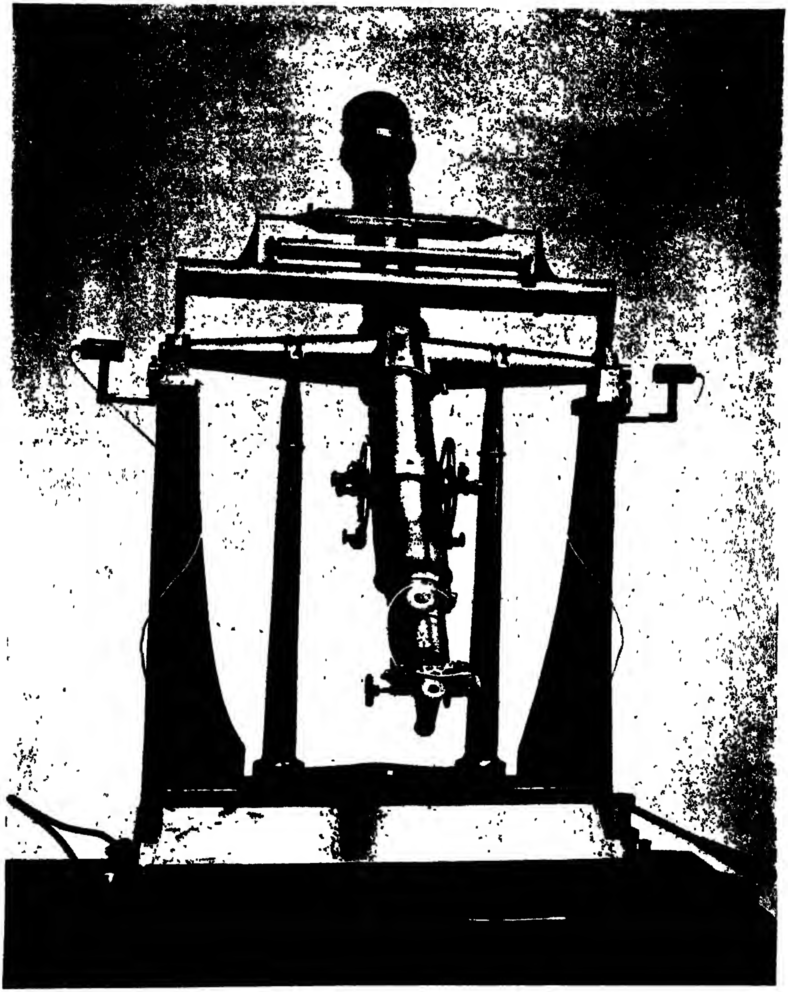

Fig. 61. Portable Transit (with transit micrometer). *(Coast and Geodetic Survey.)*

The transit instrument itself consists of a telescope with a rotation axis rigidly attached at right angles to it; this axis terminates in pivots which rest in wye bearings at the upper ends of a pair of standards. A stride level is provided for measuring the inclination of the rotation axis. The axis of collimation, which is a line through the optical center of the objective and perpendicular to the rotation axis, rotates in a vertical plane when the horizontal axis is truly level. For the purpose of determining the time the instrument may be set in any vertical plane, for example, the vertical plane through a close circumpolar star; but in this country it is used almost exclusively in the plane of the meridian.

Figure 61 shows a portable astronomical transit used for the determination of time and longitude by the Goast and Geodetic Survey. The focal length is 94 cm., the aperture 76 mm., and the magnifying power 104 diameters.

54. The Reticle.

In the old style of transit the reticle consisted of several closely spaced vertical spider threads or of lines ruled on glass, and two

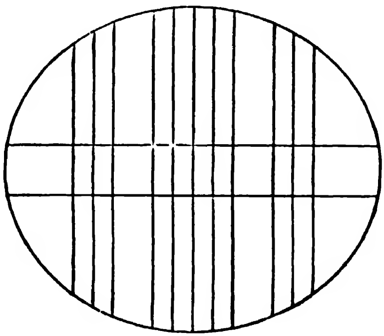

FIG. 62.

horizontal threads or lines to limit the portion of the vertical threads used for observations. A common arrangement of the

vertical threads, when the chronograph is to be used for recording the observed time, is shown in Fig. 62, the smallest intervals corresponding to about 2.5^s of time for an equatorial star.

55. Transit Micrometer.

The hand-driven transit micrometer has now replaced the old style of reticle on the instruments of the United States Coast Survey. In this instrument (Fig. 63) a single vertical thread is made to traverse the field of the telescope at such a speed that it continually bisects the star that is being observed. The record on the chronograph of the passage of the star over certain fixed points in the field is made automatically by means of an electric circuit. An automatic cut-out is so arranged as to keep the circuit broken except during four revolutions of the screw in the central part of the field. The contact points are placed so as to record twenty observations on the star, arranged in four groups. The observer has simply to set the thread on the star and follow it until it has passed beyond the range of observation. The observer does not know exactly when the observations are being made; he simply watches the thread and the star and keeps the bisection as nearly perfect as he can. It is necessary to use both hands in order to give the thread a steady motion. The result of these observations is the same as though the observer had noted accurately the time of passage of the star over 20 vertical threads. The great advantage of the instrument is that the large personal error due to estimating times of transit over the threads is almost wholly eliminated. A further advantage is that 20 observations may be made in about ten seconds, on an equatorial star, thus permitting observations on stars culminating in quick succession.

56. Illumination.

The field of the telescope is illuminated by means of a lamp or an electric bulb which sends light through the hollow axis of the instrument to a mirror at the center of the telescope, which reflects it down the telescope tube to the reticle. The threads appear as black lines against a bright field.

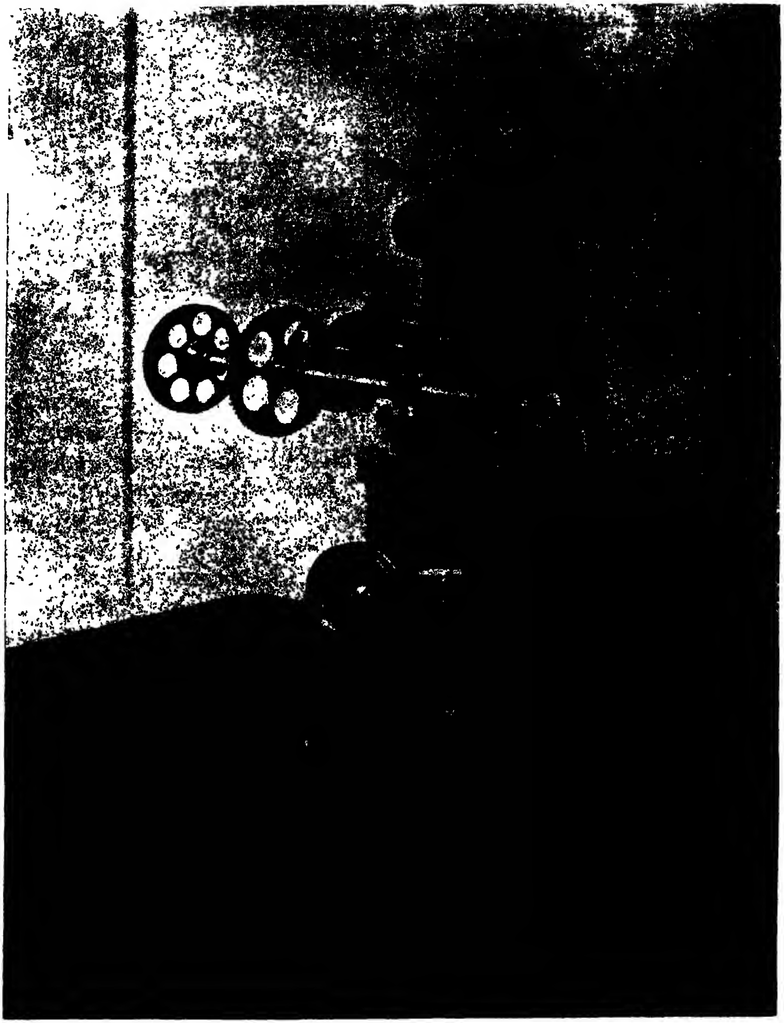

FIG. 63. The Transit Micrometer. (*Coast and Geodetic Survey.*)

56a. The Broken Telescope Transit.

During the trans-Atlantic longitude determinations of 1916 the Bamberg "broken-telescope" transit was used, and with such satisfaction that ever since that time this type of transit has been the favorite one for this kind of work. The object glass has an aperture of 7 cm. and the focal length is 67 cm. The eye-piece is mounted at one end of the (hollow) axis of the transit

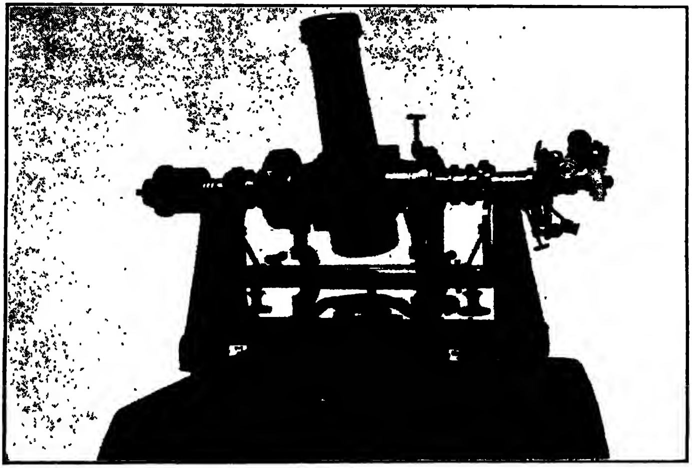

FIG. 64. The "Broken-Telescope" Transit. (*Coast and Geodetic Survey.*)

and the light passing through the objective is reflected at right angles by a prism placed at the center of the instrument. (Fig. 64.) This instrument is fitted with a micrometer similar to that shown in Fig. 63, which records at each one-tenth of a turn. Two additional contacts are placed, one each side of the center, to identify the zero mark.

At the opposite end of the axis from the eye-piece is a small electric light which illuminates the field of view. A very small prism is cemented to the larger prism on the center line of the axis, and with the faces parallel, so that the light passes directly

to the eye-piece. A small setting circle (15 cm. in diameter) is placed just back of the micrometer. This carries a vernier and level, and reads to $1'$.

A reversing apparatus enables the observer to turn the transit quickly from the direct to the reversed position. In using the transit for time observations it is customary to reverse the axis for each star observed, in the middle of the series of observations, thus eliminating the collimation error for each star separately. This necessitates two settings of the finder circle for each star. The setting is made for the first position of the axis and, before the observations are begun, and without disturbing the position of the telescope, the setting is made ready for the second position.

The cross level hangs beneath the horizontal axis instead of being placed above it as in the ordinary form of transit. Two small cross levels show when the hanging level is in the correct position to read.

The instrument is carefully counterpoised so as to avoid flexure of the telescope tube, and a portion of the weight rests on springs in the reversing apparatus, thus relieving the pivots from undue wear.

This instrument can also be used for observing latitude by the Horrebow-Talcott method. A different micrometer is used for this work, and twin levels can be attached when observing for latitude.

57. Chronograph.

The chronograph is a registering apparatus driven by clock-work, and connected electrically with a chronometer and with either the transit micrometer or an observing key. The record is made on a sheet of paper wound around a drum which revolves once per minute. A pen fastened to the armature of an electro-magnet is carried by a screw in a direction parallel to the axis of the drum. These combined motions cause the pen to draw a line spirally around the drum. When the sheet is laid flat, the record appears as a series of straight parallel lines. The chronometer breaks the circuit once per second (or two seconds),

and this break allows the armature spring to move the pen to one side and make a small notch on the record. The times of passage of stars over the threads of the transit are also recorded in a similar manner. The character of the two kinds of marks is usually dissimilar, and they may easily be distinguished. If any

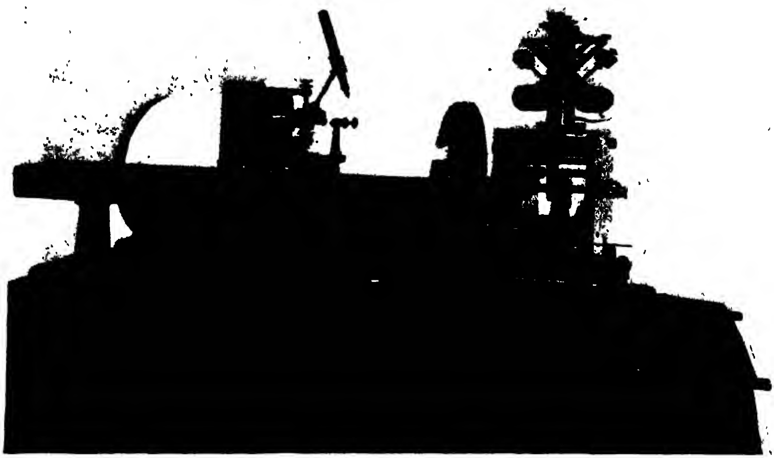

Fig. 65. Chronograph. (*Coast and Geodetic Survey.*)

one of the chronometer marks on the record sheet is identified, then the chronometer time of every mark on the sheet becomes known, and the determination of the fraction of a second for each observation is simply a matter of scaling off the position of the corresponding mark. A convenient way to mark the time without disturbing the sheet is to make notches on the sheet by means of the observing key, the number of marks so made showing the number of some minute of the chronometer reading. The speed and the diameter of the cylinder are usually such as

to make one second of time occupy a space of one centimeter. Figures 65 and 66 show chronographs such as are used in longitude observations.

Figure 69 shows a portion of a chronograph record.

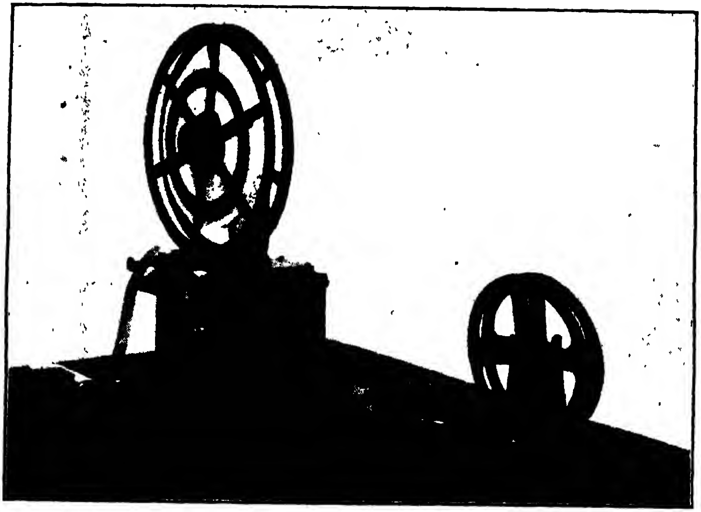

FIG. 66. Fillet type of Chronograph. (*Coast and Geodetic Survey.*)

58. Circuits.

The arrangements of circuits for operating the chronograph are shown in Figs. 67 and 68. The chronometer is placed in a separate circuit having a battery of only one cell, in order to avoid injury to the mechanism, and operates the chronograph circuit through the points of a relay. The transit micrometer operates on the *make-circuit*, which is converted into breaks by a relay. If a key is used, it replaces the micrometer relay and breaks the circuit when the key is pressed.

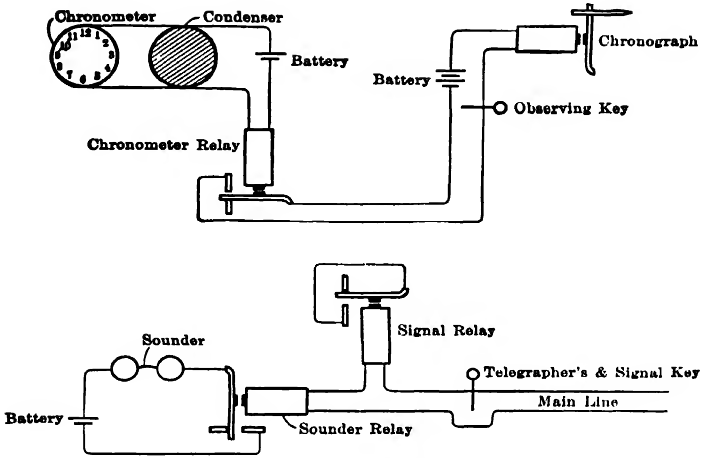

FIG. 67. Electrical Connections — Time Observations by Key Method.

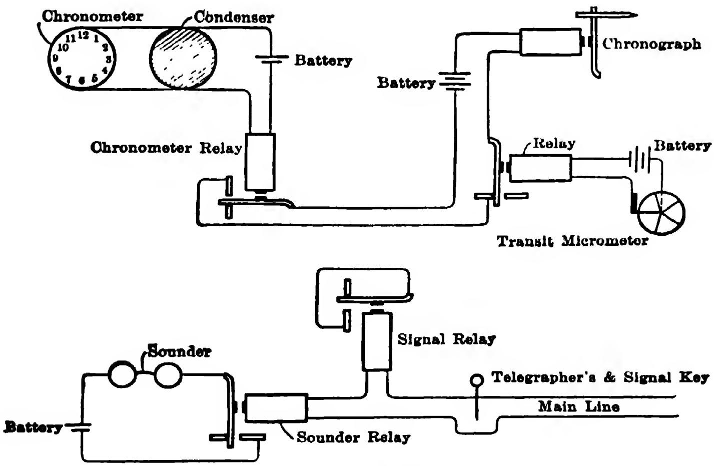

FIG. 68. Electrical Connections — Time Observations by Transit Micrometer Method.

59. Adjustment of the Transit.

In placing the transit on the supporting pier before adjusting it in the meridian, the base of the instrument must be placed so nearly in the meridian that all further adjustment in azimuth

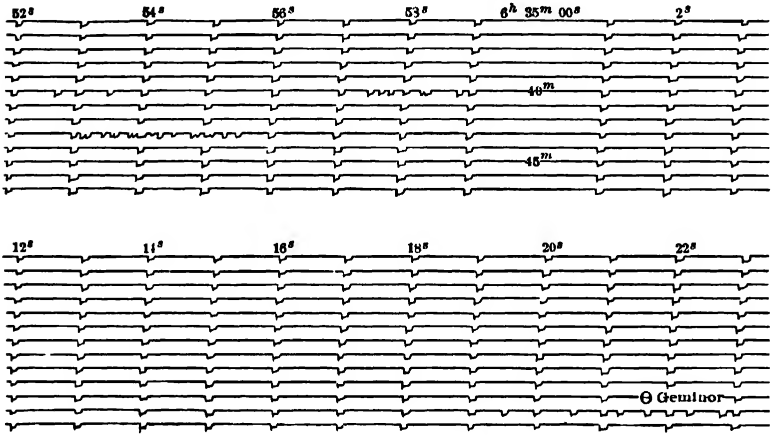

FIG. 69. Chronograph Record.

may be made by the adjusting screws provided for this purpose. The foot plates should then be cemented to the pier. The telescope is focused as in an engineer's transit — first the eyepiece, then the objective. A distant terrestrial object may be used for the first trial, but the final focusing should be done at night on the stars. A difference is usually noticed between the focus required by day and that found at night when artificial light is used.

The striding level and the horizontal axis may be adjusted simultaneously by placing the level in position, reading both ends of the bubble, then reversing it, end for end, and taking another set of readings. Half the displacement of the bubble may be corrected by adjustment of the level and half by leveling the axis.

The verticality of the threads or the micrometer line is tested by rotating the telescope slightly about its horizontal axis and noting whether a fixed object remains continuously on the thread as it traverses the field of view. Adjustment is made by rotating the diaphragm or the micrometer box until this condition is fulfilled.

The collimation is adjusted by placing the middle line of the reticle or the mean position of the micrometer line as nearly as possible in the collimation axis. To test this, point the wire on some object, reverse the telescope in its supports (axis end for end), and see if the object is still sighted. If it is not, bring the wire halfway back by means of the lateral adjusting screws.

The finder circles should be tested to see if they read zero when the collimation axis is vertical. Point on some object, level the bubble, and read the circle. Reverse the telescope, point on the same object, and repeat the readings. The mean reading is the true zenith distance, and half the difference between the two readings is the error of adjustment. Set the vernier to read the true zenith distance, sight the object again, and then center the bubble by means of the adjusting screws.

To place the line of collimation in the meridian, first determine a rough chronometer correction by leveling the axis and setting the circles for the zenith distance of some star which is near the zenith and which is about to culminate. If the (sidereal) chronometer is nearly regulated to local sidereal time, the right ascension of such a star will be nearly the same as the chronometer reading. If the chronometer is not regulated at all, it may be set approximately right by calculating the sidereal time corresponding to the mean time as indicated by a watch. An error of one or two minutes will not cause great inconvenience, as all that is necessary is to identify the star and begin observing before it has passed. The time at which this star will pass the middle vertical thread must necessarily be very close to the true sidereal time (right ascension of star), because near the zenith

the effect of the azimuth error on the observed time is very small. The difference between the right ascension of the star and the chronometer reading is an approximate value of the chronometer error. Using this value of the chronometer error, calculate the chronometer time when some slowly-moving (circumpolar) star will pass the meridian. When this calculated time arrives, point the middle thread or the micrometer thread on the star, using the azimuth adjustment screws. This places the instrument nearly in the meridian. A repetition of the whole process (on a different pair of stars) will give a still closer approximation.

It is not necessary or desirable to spend much time in reducing the errors of azimuth, level, and collimation to very small quantities. They should be so small as to cause no inconvenience in making the observations and in computing the results, but since they must be determined and allowed for in any case, the final result is quite as accurate if the errors themselves are not extremely small.

60. Selecting the Stars for Time Observations.

There are two general methods of selecting the stars to be used for a time determination. The older method requires observations on ten stars, five with the axis of the telescope in one position (say illumination or clamp *east*) and five with the axis reversed (illumination or clamp *west*). In each half-set one of the stars is a slow-moving one, that is, one situated near the pole. Of the remaining four stars in each half-set two should preferably be north of the zenith and two south of the zenith, and in such positions that their azimuth errors balance each other, that is, their A factors (see Art. 66) should add up to zero.

In the more modern method, used with the transit micrometer, twelve stars are employed, six in each position of the axis. None of these is near the pole, but their positions are so chosen as to make the algebraic sum of their A factors nearly equal to zero.

By the older method the error in azimuth adjustment is more accurately determined, but with a proper selection of stars the value of the azimuth correction need not be determined so ac-

curately, because it has a relatively small effect upon the computed chronometer correction.

In preparing for observations a list of stars should first be made out, giving the name or number of each star, its magnitude, right ascension, declination, and zenith distance, together with the *star factors* depending upon its position, as explained later. The declination of the stars chosen should be such that the algebraic sum of the A factors is less than unity. It is desirable that the list contain as many stars per hour as possible, but sufficient time must be allowed for reading the stride level, reversing the instrument, making records, etc. The telescope should be reversed before each half-set. In preparing this list the zenith distance of a star is computed by the relation

$$\zeta = \phi - \delta, \tag{17}$$

where ζ is the zenith distance (positive if south of the zenith), ϕ is the latitude, and δ is the declination (positive for stars north of the equator).

61. Making the Observations.

In beginning the observations, set the vernier of the finding circle at the zenith distance of the first star and bring the bubble to the center of its scale by moving the whole telescope. The clamp had better not be used if the telescope can be relied upon to remain in position when unclamped. When the star appears in the field, bring it between the two horizontal hairs by tapping the telescope with the finger. Set the micrometer line on the star and keep it bisected until the observations (4 turns of screw) are completed. If the instrument is not provided with a micrometer, the observer simply presses the observing key as the star passes each of the vertical threads. When the observations are made by the key method, the observer attempts to press the key as soon as possible after the star is actually bisected by the wire. In doing this he makes an error which tends to become constant as the observer gains in experience. This is known as his *personal equation*. Since the personal equation depends

chiefly upon the rapidity and uniformity with which the observer is able to record his observations, rather than upon his ability to bisect the star's image, the use of the transit micrometer very nearly eliminates this error.

After half the stars in one set have been observed, the axis should be reversed, end for end, in the supports. The striding level should be read one or more times during each half-set. If the pivots are not truly circular in section, the average inclination of the axis may be found by taking level readings with the telescope set at different zenith distances, both north and south.

The striding level should be used with great care, because the level corrections may be relatively large and cannot be eliminated by the method of observing, as in case of the collimation error and, to some extent also, the azimuth error.

Following is a record of a set of observations as read from the chronograph sheet, together with the readings of the striding level. (See United States Coast and Geodetic Survey Special Publication No. 14, p. 21.)

62. The Corrections.

The corrections that have to be applied to the mean of the observed times, to reduce it to the time corresponding to the meridian passage are those for (1) level, (2) collimation, (3) azimuth, (4) rate, and (5) diurnal aberration.

63. Level Correction.

The level correction to any observed time, Bb, is made up of the constant b, depending upon the level readings, and a factor B, depending upon the position of the star and upon the observer's latitude. If w and e are the readings of the west and east end of the level bubble in one position, and w' and e' the readings for the second position, then for the first position, the inclination of the axis of the level in terms of scale divisions is $\frac{1}{2}(w - e)$; for the second position it is $\frac{1}{2}(w' - e')$. The mean of the two is the inclination of the transit axis, free from errors of adjustment of the level. If b represents the inclination, then

Station, Key West. Date, Feb. 14, 1907. Instrument, transit No. 2, with transit micrometer. Observer, J. S. Hill. Recorder, J. S. Hill. Chronometer, Sidereal 1824.

Star: S. Monocer.	ψ^5 Aurigae	18 Monocer	ζ Geminor.	ζ Geminor.	63 Aurigae
Clamp: W	W	W	W	W	W
Level: W d: N 62 0, 17 7, +44 3 E d: 20 0, 59 5, −39 5 +4 8		W d: S 61 2, 17 7, +43 5 E d: 19 1, 59 6, −40 2 +3 3		W d: N 61 5, 17 7, +43 8 E d: 19 5, 59 7, −40 2 +3 6	
Computation of level constant. Mean N + 4 20 d; S + 3 30; + 3 75 × 0 039 = + 0 146 = b_W					
h m 6 35	h m 6 39	h m 6 42	h m 6 46	h m 6 58	h m 7 01

s	s	Sums	s	s	Sums	s	s	Sums	s	s	Sums	s	s	Sums	s	s	Sums
32 0	41 1	73 1	41 3	54 0	95 3	41 5	50 5	92 0	19 5	30 4	49 9	16 2	26 0	42 2	55 3	67 0	122 3
32 4	41 1	0 5	41 8	53 5	0 3	41 9	50 2	0 1	20 0	30 1	50 1	16 5	25 5	2 0	55 6	66 5	0 1
33 1	40 4	0 5	42 8	52 6	0 4	42 5	49 7	0 2	20 6	29 4	0 0	17 2	24 8	2 0	56 4	65 8	0 2
33 6	39 8	0 4	43 5	51 9	0 4	43 1	49 1	0 2	21 3	28 7	0 0	17 7	24 3	2 0	57 1	65 1	0 2
33 9	39 5	0 4	43 9	51 4	0 3	43 3	48 8	0 1	21 7	28 3	0 0	18 0	23 9	1 9	57 5	64 6	0 1
34 6	38 8	0 4	44 7	50 6	0 3	44 0	48 1	0 1	22 3	27 6	49 9	18 8	23 1	1 9	58 4	63 9	0 3
35 0	38 5	0 5	45 3	50 3	0 6	44 3	47 9	0 2	22 8	27 1	9 9	19 1	22 9	2 0	58.8	63 4	0 2
35 6	37 9	0 5	46 0	49 3	0 3	44 8	47 3	0 1	23 6	26 4	50 0	19 8	22 3	2 1	59 5	62.6	0 1
36 1	37 4	0 5	46 9	48 5	0 4	45 4	46 6	0 0	24 3	25 7	0 0	20 5	21 6	2 1	60 3	61 9	0 2
36 4	37 1	0 5	47 2	48 1	0 3	45 7	46 3	0 0	24 6	25 4	0 0	20 7	21 4	2 1	60 7	61 5	0 2
		Sum 734 6			Sum 953 6			Sum 921 0			Sum 499 8			Sum 420 3			Sum 1221 9

Mean	36 73	47 68	46 05	24 99	21 02	01 10
R*						
κ	− 0 02	− 0.03	− 0 02	− 0 02	− 0 02	− 0 02
Bb	+ 0 14	+ 0.19	+ 0 14	+ 0 17	+ 0 16	+ 0 18
t	6 35 36 85	6 39 47 84	6 42 46 17	6 46 25 14	6 58 21 16	7 05 01 26
α	6 35 51.85	6 40 02 92	6 43 01 21	6 46 40 17	6 58 36 16	7 05 16 28
$(\alpha - t)$	+15.00	+15 08	+15 04	+15 03	+15 00	+15 02

* R, correction for rate, is negligible in this time set.

$$b = \tfrac{1}{2}\left[\tfrac{1}{2}(w - e) + \tfrac{1}{2}(w' - e')\right]$$
$$= \tfrac{1}{4}\left[(w + w') - (e + e')\right].$$

If d is the value of one division of the level scale expressed in seconds of arc, then b in seconds of time is

$$b = \frac{d}{60}\left[(w + w') - (e + e')\right], \qquad [18]$$

in which the scale divisions are supposed to be numbered each way from zero; b is positive if the west end of the axis is too high. If, however, the divisions of the level are numbered continuously from one end of the tube to the other, the equation is

$$b = \frac{d}{60}\left[(w - w') + (e - e')\right], \qquad [19]$$

in which the primed letters refer to that position of the level in which the zero of the scale is west.

64. Pivot Inequality.

If the pivots are found to be unequal in diameter, then the apparent inclination as found from the level readings must be corrected by a quantity p, which is the inequality as found by a special set of readings of the level. If β_e and β_w are the inclinations as derived from the level readings, and b_e and b_w the true inclinations for the two positions of the axis,

then
$$p = \frac{\beta_e - \beta_w}{4};$$

also and
$$\left.\begin{aligned} b_e &= \beta_e - p \\ b_w &= \beta_w + p. \end{aligned}\right\} \text{for "clamp west."} \qquad [20]$$

To determine the effect of this inclination error on the observed time of transit of any star, let S (Fig. 70) be the star observed, and let HS be the path of the vertical thread, inclined to the true vertical at an angle b. In the triangle PHS the angle at P is the error which is to be computed. The angle at H is b; PS is the polar distance, or $90° - \delta$; HS is the altitude (nearly), or $90° - \zeta$. From the triangle PHS,

$$\frac{\sin P}{\sin H} = \frac{\sin HS}{\sin PS},$$

or

$$P = b \cos \zeta \sec \delta \text{ (approx.)}$$
$$= b \cdot B. \qquad [21]$$

The factor B may be taken from Table III when the zenith distance and the declination of the star are known.

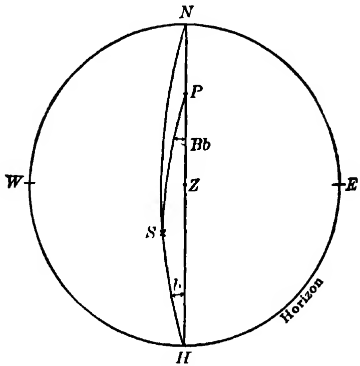

FIG. 70.

65. Collimation Correction.

The correction to the observed time is cC, c being the constant angle between the collimation axis and the *mean thread*, expressed

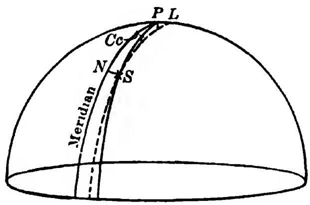

FIG. 71.

in seconds of time, and C the collimation factor, varying with the position of the star. The collimation constant c may be found

by special observations, but is usually computed from the time observations themselves, as explained later; it is considered positive if the line of sight is east of the true position when the clamp is east.

In Fig. 71, P is the pole, S the star, PN the meridian, and SL the trace of the thread all points of which are at the same distance (c) from PN. The error is the angle P. Since the angle N is 90°,

$$\sin P = \frac{\sin SN}{\sin PS} = \frac{\sin c}{\cos \delta},$$

or

$$P = c \sec \delta = cC. \qquad [22]$$

The collimation factor C will be found in Table III.

66. Azimuth Correction.

The error of setting the instrument in the meridian is measured by the constant a, the azimuth of the axis of collimation expressed

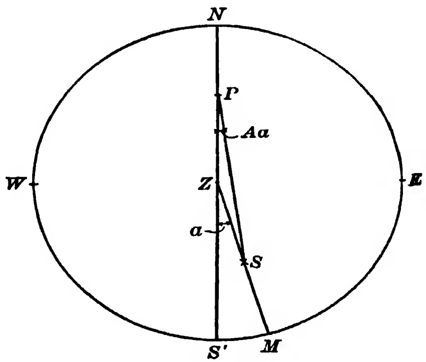

FIG. 72.

in seconds of time. This constant is derived from the variations in the observations themselves. In Fig. 72, P is the pole, Z the zenith, and S the star. In the triangle PZS, P is the required correction, and $S'ZS$ is a, the azimuth error. Applying the law of sines,

$$\frac{\sin P}{\sin S'ZS} = \frac{\sin \zeta}{\cos \delta},$$

or

$$P = a \sin \zeta \sec \delta$$
$$= a \cdot A. \qquad [23]$$

The azimuth factor A may be taken from Table III. The constant a is positive when the plane of the axis of collimation is east of south. A is positive for all stars except those between the zenith and the pole.

67. Rate Correction.

In order to compute these corrections it is necessary to reduce all observations of the chronometer correction to some definite epoch, for example, the mean of all the observed times, so that variations in the chronometer correction itself will not affect the determination of the transit errors. This is done by applying the correction

$$R = (t - T_o)\, r_h, \qquad [24]$$

where t is the chronometer time of transit,

T_o is the mean epoch of the set,

and r_h is the hourly rate of the chronometer, positive if losing, negative if gaining.

68. Diurnal Aberration.

The motion of the observer due to the diurnal motion of the earth makes all stars appear farther east than they actually are; in other words it apparently increases their right ascensions. The amount of the correction is expressed by the equation

$$\kappa = 0^s.021 \cos \phi \sec \delta. \qquad [25]$$

This formula may be derived as follows: the velocity of a point on the earth's equator (toward the east) is 0.288 mile per second. For any other latitude the velocity is 0.288 cos ϕ mile per second. The velocity of light is 186,000 miles per second, and the angular displacement (κ') of the star toward the east point of the horizon is therefore equal to $\tan^{-1}\dfrac{0.288 \cos \phi}{186{,}000}$. The effect on the ob-

served time is the angle κ at the pole, Fig. 73. Hence

$$\frac{\sin \kappa}{\sin 90^\circ} = \frac{\sin \kappa'}{\cos \delta}$$

or

$$\kappa = 0''.319 \cos \phi \sec \delta$$
$$= 0^s.021 \cos \phi \sec \delta.$$

Values of this correction will be found in Table IV.

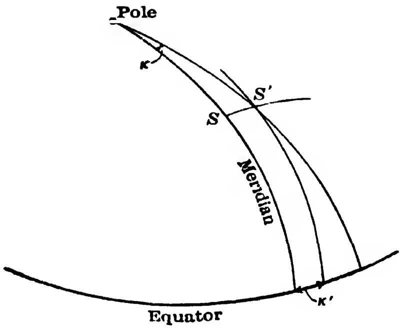

FIG. 73.

69. Formula for the Chronometer Correction.

The true sidereal time, or right ascension of the star, is given by the equation

$$\alpha = t + \Delta T + \kappa + R + Aa + Bb + Cc, \qquad [26]$$

in which t is the mean of the observed transits and ΔT is the chronometer correction. Since the corrections for aberration, rate, and inclination may be found directly, they are applied to t at once. If we call t_1 the value of t thus corrected, then

$$\alpha - t_1 = \Delta T + Aa + Cc,$$

or

$$\Delta T = (\alpha - t_1) - Aa - Cc. \qquad [27]$$

70. Method of Deriving Constants a and c, and the Chronometer Correction, ΔT.

The method shown in the following table is the one used when the observations are made with the transit micrometer and

when the latitude is less than 50°. For greater latitudes the observations are reduced by the method of least squares.

COMPUTATION OF TIME SET.

[Station, Key West, Florida. Date, Feb. 14, 1907. Set, 2. Observer, J. S Hill. Computor, J. S. Hill.]

Star.	Clamp.	$\alpha - t$	δt	C	A	Cc	Aa	$\Delta T = (\alpha - t) - Cc - Aa$	v
		s	s			s	s	s	s
1 S Monocer	W	+15 00	0 00	+1 02	+0 26	+0 27	+0 02	+14 71	+0 02
2 ψ^6 Aurigae	W	+15 08	+0 08	+1 38	−0 15	+0 36	−0 03	+14 75	−0 02
3 13 Monocer..	W	+15 04	+0 04	+1 01	+0 37	+0 26	+0 03	+14 75	−0 02
4 θ Geminor	W	+15 03	+0 03	+1 21	−0 20	+0 32	−0 01	+14 72	+0 01
5 ζ Geminor..	W	+15 00	0 00	+1 07	+0 07	+0 28	0 00	+14 72	+0 01
6. 63 Aurigae	W	+15 02	+0 02	+1 30	−0 [illegible]	+0 34	−0 02	+14 70	+0 03
7 ι Geminor	E	+14 43	−0 57	−1 13	−0 07	−0 30	0 00	+14 73	0 00
8. β Can. Min.	E	+14 45	−0 55	−1 02	+0 28	−0 27	+0 01	+14 71	+0 02
9. α Can Min	E	+14 45	−0 55	−1 01	+0 33	−0 26	+0 01	+14 70	+0 03
10 β Geminor	E	+14 41	−0 59	−1 13	−0 08	−0 30	0 00	+14 71	+0 02
11 π Geminor.	E	+14 42	−0 58	−1 21	−0 19	−0 32	−0 01	+14 75	−0 02
12 ϕ Geminor.	E	+14 47	−0 53	−1 12	−0 05	−0 29	0 00	+14 76	−0 03

Mean $\Delta T = + 14\ 727$

$3\ 00\ \delta t + 3\ 10\ c + 0\ 70\ a_W - 0\ 04 = 0$

$3\ 00\ \delta t + 3\ 89\ c - 0\ 90\ a_W - 0\ 13 = 0$

$2\ 12\ \delta t + 2\ 75\ c - 0\ 70\ a_W - 0\ 09 = 0$ (2) × 0 707

$5\ 12\ \delta t + 5\ 85\ c - 0\ 13 = 0$ (1) + (5)

$4\ 71\ \delta t + 5\ 38\ c - 0\ 12 = 0$ (6) × 0 920

$9\ 53\ \delta t + 2\ 61 = 0$ (8) + (9) 11. $\delta t = - 0\ 274$ from (10)

$\Delta T = + 15\ 00 - 0\ 274 = + 14\ 726$

3. $3\ 00\ \delta t - 3\ 15\ c + 0\ 56\ a_E + 1\ 63 = 0$

4. $3\ 00\ \delta t - 3\ 47\ c - 0\ 34\ a_E + 1\ 74 = 0$

7 $1\ 82\ \delta t - 1\ 91\ c + 0.34\ a_E + 0\ 99 = 0$ (3) × 0 607

8 $4\ 82\ \delta t - 5\ 38\ c + 2\ 73 = 0$ (4) + (7)

12 $-1\ 32 - 5\ 38\ c + 2\ 73 = 0$ from (8) 13. $c = + 0\ 262$ from (12)

14. $-0\ 82 + 1\ 02 - 0\ 99\ a_W - 0\ 13 = 0$ 15 $a_W = + 0\ 071$

16. $-0\ 82 - 0\ 83 + 0\ 56\ a_E + 1\ 63 = 0$ 17. $a_E = + 0\ 036$

The serial numbers in the lower part of the table show the order of the different steps of the computation. Equation 1 is obtained by taking the terms corresponding to the three southernmost stars (that is, Nos. 1, 3, and 5), substituting the sums of these numbers in the equation $\Delta T + Cc + Aa - (\alpha - t_1) = 0$, and treating this result as though it were the equation for a single

TRANSIT RECORD FOR TIME.

Station, Middlesex Fells. Date, Nov. 15, 1898. Observer, G. L. H. Chronometer, Bond 541.

	π Pegasi. Clamp. *E.*	7 Lacertae. *E.*	10 Lacertae. *E*	ι Cephei. *E*	π Cephei. *W.*	Br. 3077. *W.*	τ Pegasi. *W.*	ι Androm. *W.*
Level	32 0 56 0	12 5	48 5 39 0	Temp.	59 0 30 0	22 0	32 0 56 0	
N	62 5 26 0	15 0	S 46 0 40 5	45° F.	N 41 0 48 0	9 0	S 61 0 28 0	
	94 5 82 0	13 75 × 0 0092	94 5 79 5		100 0 78 0	15 5 × 0 0092	93 0 84 0	
	82.0	= +0.13	79 5		28 0	= +0 14	81 0	
	12 5	b_e = 0 13 − 0 02 = +0 11	15 0		22 0	b_w = 0 14 + 0 02 = +0 16	9 0	
	h m s							
V	22 03 42·10	22 25 10 71	22 32 53 50	22 43 45 90	I 23 01 46 64	23 06 21 74	23 13 53.08	23 32 19.15
	56 79	29 95	33 09 41	44 15 82	02 34 00	43 82	14 06 64	35 84
	04 11 45	46 85	25 12	45 96	03 21 63	07 06 40	20 15	52 30
	26 10	26 08 42	41 07	45 15 85	04 08 95	28 67	33 70	33 09 52
I	40 96	27 25	57 05	45 82	V 04 56 00	51 40	47 10	26.42
	22 04 11 48	22 25 49 04	22 33 25 23	22 44 45 87	23 03 21 44	23 07 06 41	23 14 20 13	23 32 52 65
κ	−0 02	−0 02	−0 02	−0 04	−0 06	−0 03	−0 02	−0 02
Bb	+0 13	+0 17	+0 14	+0 25	+0 52	+0 28	+0 16	+0 22
Rate	−0 03	−0 01	−0 01	0 00	+0 01	+0 01	+0 01	+0 02
t	22 04 11 56	22 25 49 18	22 33 25 34	22 44 46 08	23 03 21 91	23 07 06 67	23 14 20 28	23 32 52 87
α	22 05 30 65	22 27 08 45	22 34 44 50	22 46 05 99	23 04 42 76	23 08 26 16	23 15 39 26	23 33 12 14
α − t	+01 19.09	1 19 27	1 19 16	1 19 91	1 20 85	1 19.49	1 18 98	+ 1 19 27

$\frac{d}{60} = 0\ 0092$ $r_h = +0\ 033.$ $p = +0\ 02$

FIG. 74.

COMPUTATION OF ΔT

Star.	Pos.	$\alpha - t_1$.	C.	A.	Cc	Aa	ΔT. $\alpha - t_1$, $-Cc-Aa$.	v	
π Pegasi	E	+1 − 19 09	−1 19	+0 20	−0 02	−0 15	+1 19 26	−0 05	
7 Lacertae	E	19 27	−1 55	−0 20	−0 02	+0 15	19 14	+0 07	
10 Lacertae .	E	19 16	−1 28	+0 09	−0 02	−0 07	19 25	−0 04	
ι Cephei . ..	E	19 91	−2 43	−0 96	−0 04	+0 74	19 21	0 00	$\Delta T = +1^m\ 19^s\ 21$ at $22^h\ 52^m$
π Cephei	W	+1 20 85	+3.82	−2 05	+0 06	+1 58	19 21	−0 01	
Br. 3077	W	19 49	+1 81	−0 44	+0 03	+0 34	19 12	+0 08	
τ Pegasi	W	18 98	+1 09	+0 36	+0 02	−0 28	19 24	−0 04	
ι Androm..	W	19 27	+1 36	−0 01	+0 02	+0 01	19 24	−0 04	
		$\alpha - t_1$	C	A.	Cc	$\alpha - t_1$, $-Cc$	Aa.	$\alpha - t_1$, $-Cc$ $-Aa$	$c = +0.03$
1st. approx. Mean of time stars	E	1 − 19 17	−1 34	+0 0[illegible]	−0 04	19 21	−0 02	+1 19 23	$a_E = -0\ 78$
1st. approx. Azimuth star	E	1 − 19 91	−2 43	−0 96	−0 07	19 98	+0 75	19 23	
1st. approx. Mean of time stars	W	1 − 19 25	+1 42	−0 03	+0 04	19 21	+0 02	19 19	$a_W = -0\ 76$
1st. approx. Azimuth star	W	1 − 20 85	+3 82	−2 05	+0 11	20 74	+1 56	19 18	
2d. approx. Mean of time stars					−0 02	19 19	−0 02	+1 19 21	$c = +0\ 016$
2d. approx. Azimuth star					−0 04	19 95	+0 74	19 21	$a_E = -0.768$
2d. approx. Mean of time stars					+0 02	19 23	+0 02	19 21	$a_W = -0\ 772$
2d. approx. Azimuth star..					+0 06	20 79	+1.58	19 21	

FIG. 74.

star. Equations 2, 3, and 4 are found in a similar manner. This gives four equations for the twelve stars, two for each half-set. Since there are now as many equations as there are unknowns, the quantities c, a_W, a_E, and ΔT may be found by solving these equations simultaneously. Notice that in this solution 15^s has been dropped from ΔT, and that δt is the small correction which must be added to 15^s to obtain ΔT.

The following method of deriving the constants and the chronometer correction without employing least squares is applicable when the two groups of stars have A factors which are not so nearly balanced, or where the list of observed stars consists of one slowly-moving (azimuth) star and several time stars in each half-set. This method gives, by a series of approximations, very nearly the same result that would be obtained by the method of least squares. The various steps in the computation are shown in tabular form in Fig. 74.

The formulas on which the method is based are as follows: For each star we may write an equation of the form

$$\alpha - t_1 = \Delta T + Aa + Cc. \tag{28}$$

Then for the east and west groups we have

$$\left.\begin{aligned}(\alpha - t_1)_W &= \Delta T + A_W a_W + C_W c,\\ (\alpha - t_1)_E &= \Delta T + A_E a_E + C_E c.\end{aligned}\right\} \tag{a}$$

Assuming at first that a_E and a_W are equal, we find an approximate value of c by subtracting the second equation from the first. Solving for c, we find

$$c = \frac{(\alpha - t_1)_W - (\alpha - t_1)_E}{C_W - C_E}.$$

In the above example,

$$c = \frac{19.25 - 19.17}{1.42 + 1.34} = +0.03.$$

Using this approximate value of c, the last terms in Equations (*a*) are computed and subtracted from $(\alpha - t_1)$ in each case, leaving

the equations in the form

$$(\alpha - t_1 - C_c) = \Delta T + A_W a_W.$$

Taking each half-set separately, and also grouping the azimuth star and the time stars separately, we have for the next group

$$\left.\begin{aligned}(\alpha - t_1 - C_c) &= \Delta T + (A_W a_W)\ az,\\ (\alpha - t_1 - C_c) &= \Delta T + (A_W a_W)\ time,\end{aligned}\right\} \qquad (b)$$

and a similar pair of equations for the second position of the axis. From Equations (b) we derive

$$a_W = \frac{(\alpha - t_1 - C_c)_{az} - (\alpha - t_1 - C_c)_{time}}{A_{az} - A_{time}}.$$

In the example,

$$a_E = \frac{19.98 - 19.21}{-0.96 - 0.03} = -0.78$$

and
$$a_W = \frac{20.74 - 19.21}{-2.05 + 0.03} = -0.76.$$

Employing these approximate values of a_E and a_W, the Aa corrections are computed and subtracted, giving the value in the column headed $\alpha - t_1 - C_c - Aa$. For the time stars these values are 19.23 and 19.19. Since these values do not agree for the two positions of the instrument, the value of c is evidently in error. A second approximation must be made by treating the difference of these numbers (0.04) as an error in c and obtaining a correction to c by the same process that was used in finding c in the first instance, that is,

$$\text{Correction to } c = \frac{19.19 - 19.23}{1.42 + 1.34} = -0.014.$$

Hence
$$c = +0.03 - 0.014 = +0.016.$$

With this improved value of c new values of a_E and a_W are computed as before. The second values are $a_E = -0.768$ and $a_W = -0.772$. Using these values, the chronometer corrections are found to agree, and hence no further approximation is necessary. The azimuth and collimation corrections are now found for each

star, as shown in the upper part of the table. The mean of the ΔT's for all the stars is the chronometer correction for the mean of the observed times. The residuals (v) are computed by subtracting ΔT for each star from the mean of the ΔT's for that group. These should add up nearly to zero.

Whenever the most accurate results are desired, the computation may be made by the method of least squares. For the details of this method see Coast and Geodetic Survey Special Publication No. 14, page 41.

71. Accuracy of Results.

The error in the computed value of ΔT due to accidental errors may easily be kept within a few hundredths of a second. Observations made by the key method may be subject to a large constant error, the observer's personal equation, which may be several times as large as the accidental error. Observations made with the transit micrometer are nearly free from personal errors.

72. Determination of Differences in Longitude.

The determination of the difference in longitude of two stations consists in measuring the difference between the local sidereal times at the two places, usually determined by transits of stars over the meridian. Previous to 1922 the method almost exclusively used in this country for accurate longitudes in places where a telegraph line was available was that in which the local sidereal times are compared by electric signals sent over the telegraph line.

According to the usual program each observer, provided with a transit, chronometer, and chronograph, determines the local sidereal time by the method previously described; then the two chronometers are directly compared by means of arbitrary signals, which are sent over the telegraph line and recorded simultaneously on the two chronographs; and finally, each observer again determines the local time by star transits.

According to the Coast Survey instructions (Special Publication No. 14) each half-set should consist of observations on from 5 to 7 stars (preferably 6), all of which are time stars, no

azimuth star being used in this method. The algebraic sum of the azimuth factors (A) should be less than unity. Four half-sets are observed during an evening, and the telescope axis is reversed before each half-set. The observers do not exchange places, during the occupancy of the stations, as was formerly the practice, since the transit micrometers eliminate most of the personal error. Observations on three or four sights usually give the required accuracy.

Figure 75 shows the switch-board and the arrangement of the electrical circuits required in longitude observations. When the observer is making observations for time the circuit is arranged as shown in Fig. 68. Figure 76 shows the circuits as arranged during the exchange of arbitrary signals. These signals are made by tapping the signal key in the main-line circuit. Half of these signals are sent by the eastern observer, and half by the western observer; the error due to the transmission time is eliminated by taking the mean. The chronometers mark the record sheets while the signals are being sent, so that the time of each signal may be read from each chronograph sheet. These chronometer times are reduced to true local sidereal times by applying the interpolated chronometer corrections. The difference in longitude is the difference between the true sidereal times.

73. Observations by Key Method.

If the transit micrometer is not used, the selection of stars must be modified so as to allow more time between observations. Since the observed times will be subject to the personal errors of the observers, it is important that the observers exchange places at the middle of the series, so that their relative personal equation will enter the latter half of the observations with its algebraic sign changed. The arrangement of the circuits is shown in Figs. 67 and 77, in which an observing key replaces the relay and circuit of the transit micrometer.

73a. Longitude by Radio Time Signals.

Since 1922 the difference of longitude has been determined by recording on the chronograph of the observing station the 10 P.M.

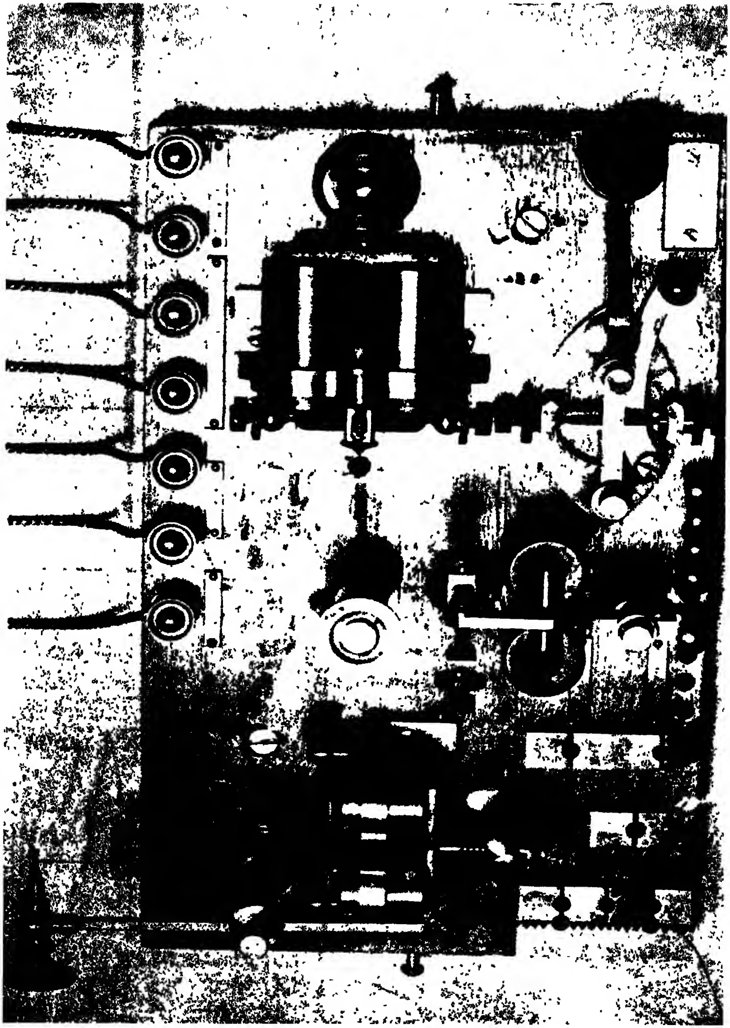

FIG. 75. Switch Board — Telegraphic Longitude. *(Coast and Geodetic Survey.)*

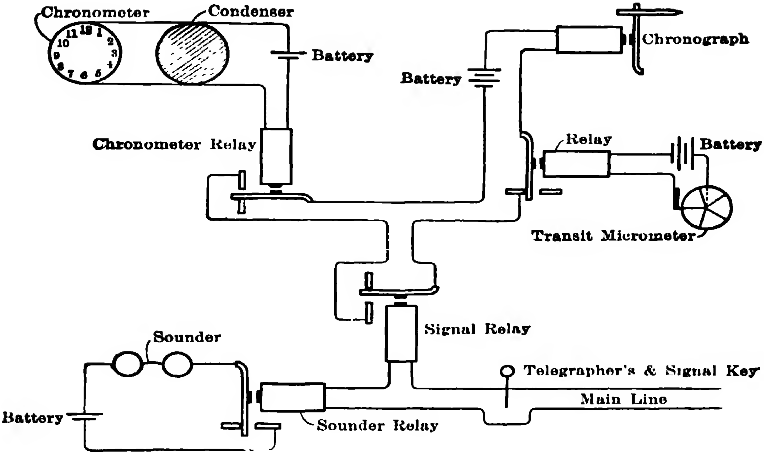

FIG. 76. Electrical Connections — Exchange of Signals — Transit Micrometer Method

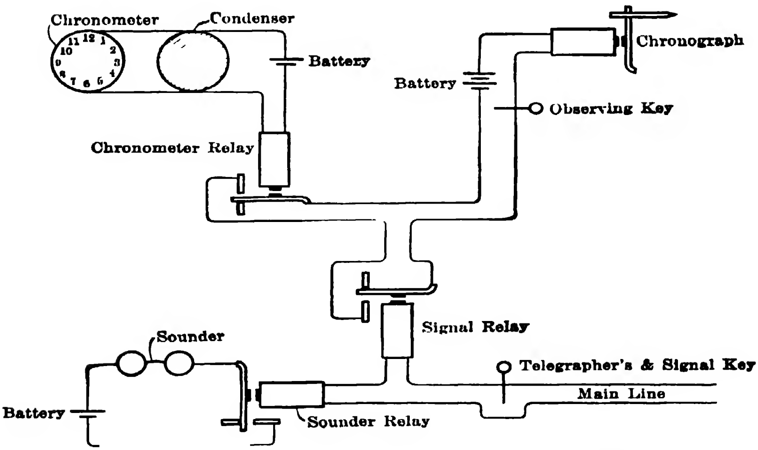

FIG. 77. Electrical Connections — Exchange of Signals — Key Method.

time signal (Eastern Time) sent out by the U. S. Naval Observatory at Washington, either over NSS (Annapolis) or NAA (Arlington). It is therefore unnecessary to occupy two longitude stations, as was formerly done, because in all cases the Naval Observatory replaces one of them. The receiving apparatus used for recording this signal was developed at the U. S. Bureau of Standards for the U. S. Coast and Geodetic Survey in 1921. The radio apparatus consists of a three-stage radio amplifier and a radio recorder. The latter has a high resistance radio relay which operates the circuit of the chronograph. Figure 78 shows the wiring diagram of the hook up for longitude work. The original apparatus is described in Special Publication No. 109, by George D. Cowie, Geodetic Engineer, Coast and Geodetic Survey, and E. A. Eckhardt, Physicist, Bureau of Standards.

The star list used and the method of making the observations are nearly the same as those previously given. The lag of the signal, which includes that of the local circuits, must be determined for any apparatus, but once determined appears to be very nearly constant. The time signal gives the instant of 22^h Eastern Standard Time. This is reduced to Sidereal time at the center of the clock house of the Naval Observatory, longitude $5^h\ 08^m\ 15^s.784$. All time signals are received at the observatory so that any error in the time of sending the signal is measured and recorded. The effect of this error may be included in the computation of the longitude. About 40 radio signals are used in comparing the local time with Washington time. The accuracy required can be obtained from about 3 or 4 nights observations.

This method eliminates one observing station and is therefore less expensive than the older method. The observer is independent of the location of telegraph lines and may determine a longitude at any place to which the instruments can be transported. The total time required is materially less than with the older method. Another advantage of this method is that since all stations are compared directly with Washington errors

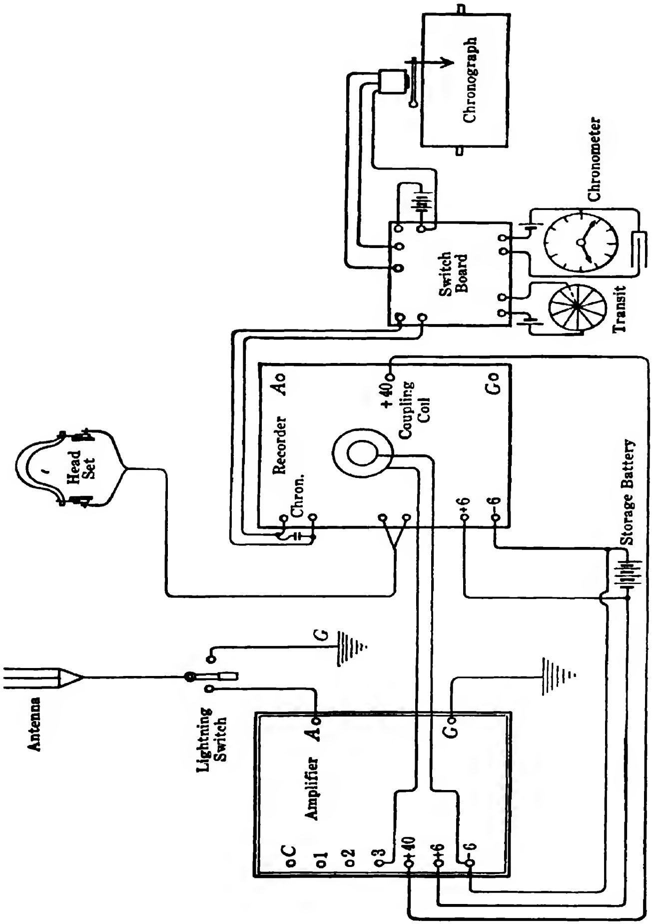

FIG. 78. Wiring Diagram of Hook-up for Longitude Work

cannot accumulate from station to station. The accuracy of the results appears to be quite as high as with the older methods.

In all longitude determinations one of the largest errors is the uncertainty in the chronometer correction. This is liable to vary irregularly, and even during the short interval between the two sets of time observations it may appreciably affect the time signals between the longitude stations. It has been proposed to use a half-seconds invariable pendulum to control the chronometer error. (See Chapter IX.) The variations in the period of these pendulums is found to be very much smaller than the variations in the running of a chronometer and this method promises to increase greatly the accuracy of longitude determinations. The chronometer can be compared with the pendulum by the method of coincidences, so it is possible to obtain a complete record of the variations in the chronometer rate during the longitude observations.

74. Correction for Variation of the Pole.

The periodic variation of the position of the pole affects all observations for longitude and must be allowed for by applying the corrections for the date of the observation. (See Art. 81, p. 149.)

74a. Accuracy of Results.

The accuracy of longitude determinations has been greatly increased by the introduction of the transit micrometer. As an example of results obtained with the broken telescope transit equipped with transit micrometer we may quote the results obtained in 1916* in connection with the trans-Atlantic longitudes, which are as follows:

Longitude differences.	Probable error of determination.
Far Rockaway — Washington	$\pm 0^s.0028$
Cambridge — Washington	$\pm 0^s.0024$
Cambridge — Far Rockaway	$\pm 0^s.0031$

* U. S. Coast and Geodetic Survey Special Publication No. 35. Observations by Fremont Morse and O. B French.

The longitudes of stations in the United States as determined from Europe through cables previous to 1900 were subject to errors of some $0^s.05$. Subsequent improvements in the methods and instruments gave results showing errors of only about one-tenth of this amount.

The accuracy of longitudes determined by radio signals appears to be about as high as that of the telegraphic determinations. The uncertain factor in these radio determinations is the amount of lag in the mechanical parts of the apparatus. The amount of this lag has been determined and it is considered that radio longitudes are sufficiently accurate for geodetic purposes.

75. Determination of Latitude.

The method which has been chiefly used in this country for determining astronomical latitudes for geodetic purposes is that known as Talcott's (or the Harrebow-Talcott) Method. The instrument employed is the zenith telescope, illustrated in Fig. 79. The principle involved is that of measuring, not the absolute zenith distances of stars, as is done with the meridian circle, but the small *difference* between the zenith distances of two stars which are on opposite sides of the zenith. By a proper selection of stars this difference in zenith distance may be made so small that the whole angular distance to be measured comes within the range of the eye-piece micrometer, which for most instruments is about half a degree. A sensitive spirit level attached to the telescope serves to measure any slight change in the inclination of the vertical axis of the instrument between the two observations on a pair of stars. The accuracy of the results obtained by this method is superior to that of every other field method, and compares favorably with the results obtained with the largest instruments.

The horizontal axis of the telescope is very short as compared with that of the transit instrument; small errors in the inclination of the axis, however, have very little effect upon the results; a close adjustment is therefore unnecessary. Since the instru-

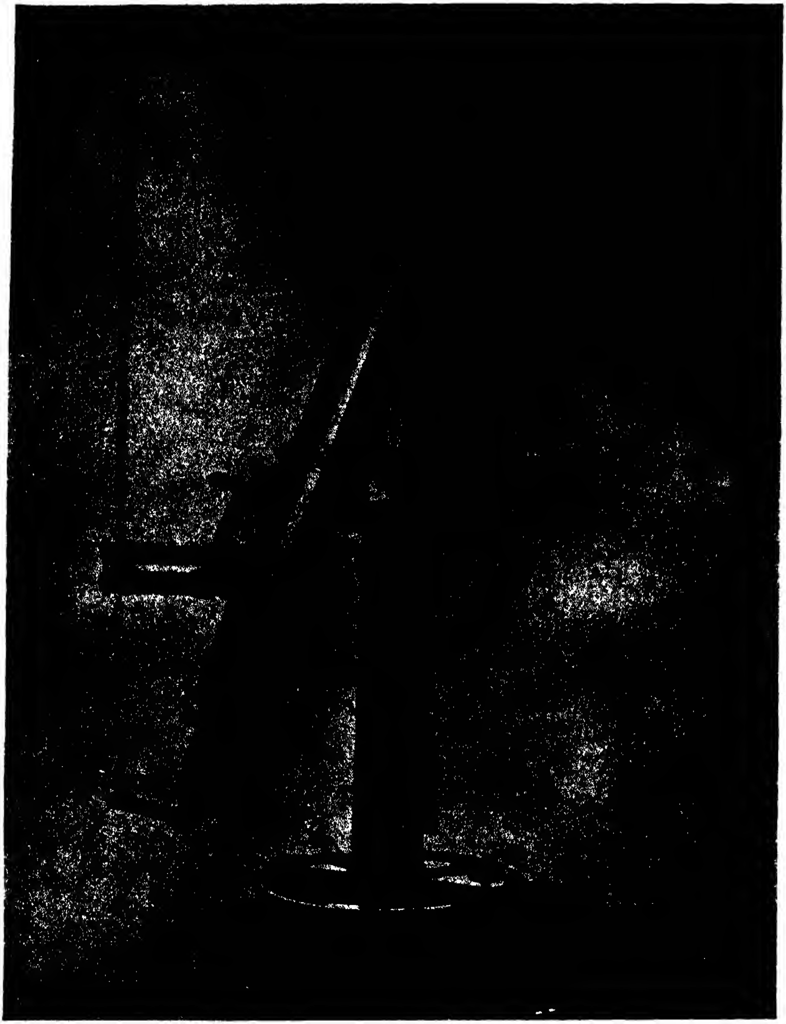

FIG. 79. Zenith Telescope. (*Coast and Geodetic Survey.*)

ment is used in the plane of the meridian and must be quickly turned from the north side to the south, or vice versa, the horizontal circle is provided with stops which are adjustable, so that the telescope may be quickly changed from one side of the zenith to the other. The micrometer, placed in the focal plane of the eye-piece, is set so as to permit of measuring small angles in the vertical plane. The head of the screw is graduated to read to about $0''.5$ directly and to $0''.05$ by estimation. The spirit level has an angular value of one (2^{mm}) division equal to about $1''.5$.

76. Adjustments of the Zenith Telescope.

When the instrument is in perfect adjustment, the plate levels should be central in all azimuths as the telescope is turned about the vertical axis. The leveling may be perfected by use of the more sensitive latitude level. The horizontal axis must be at right angles to the vertical axis. The movable micrometer threads must be truly horizontal. They may be adjusted by a method similar to that used in adjusting the engineer's level — by swinging the telescope horizontally through a small angle and observing whether the thread remains on a fixed point. The collimation adjustment should be made in the same manner as in a transit, but is not of so great importance. Allowance must be made for the eccentricity of the telescope when making the collimation adjustment. The value of one turn of the micrometer may be determined approximately by observations upon a close circumpolar star near its elongation. The most satisfactory way, however, is to derive the value of one turn from the latitude observations themselves, by the method of least squares. The value of one division of the latitude level may be determined by means of a level trier, or it may be found by varying the inclination of the telescope and employing the eye-piece micrometer to determine the amount of this inclination by observations on a terrestrial mark.

When in use the instrument is mounted on a wooden or concrete pier. It is usually protected by a tent or other temporary shelter.

In order to make the observations, it is necessary to have a

chronometer regulated to local sidereal time with an error not exceeding one second of time.

77. Selecting Stars.

The list of stars in the American Ephemeris will not ordinarily be sufficient for latitude observations, on account of the exacting nature of the conditions. It will be necessary to consult such star catalogues as Boss's *Preliminary General Catalogue of 6188 stars for the Epoch 1900*, or one of the Greenwich catalogues. In order to keep the zenith distances within the required limits, it will often be necessary to observe on stars which are much fainter than those used for time observations. The pairs of stars selected should, if possible, differ by less than 20^m in their right ascension and by less than $20'$ in their declinations. The actual zenith distance of a star should not exceed $45°$. Following is a specimen star list for zenith telescope observations.

OBSERVING LIST (FORM 1).

[St. Anne, Ill., June 25, 1908. Zenith telescope No. 4. $\phi = 41° 01'.3$. Search factor $= 2\phi = 82° 03'$.]

Star No. Boss catalogue.	Mag.	Right ascension.	Declination δ.	Difference between δ's.	$\Sigma\delta$ = sum of declinations.	$\Sigma\delta - 2\phi$.	$N - S = a$* $(\Sigma\delta - 2\phi)$.	Star north or south.	Setting = ½ difference of δ's.	Turns.
		h m s	° ′	° ′	° ′	′			° ′	
4327	4.5	16 55 22	82 11					N		12
4379	4.9	17 11 53	−0 21	82 32	81 50	−13	−17	S	41 16	28
4441	5.9	17 28 13	28 28					S		10
4494	5.8	17 42 04	53 50	25 22	82 18	+15	+20	N	12 41	30
4623	5.1	18 13 22	64 22					N		24
4651	5.4	18 18 45	17 47	46 35	82 09	+ 6	+ 8	S	23 18	16
4669	5.9	18 22 26	29 47					S		20
4711	5.5	18 31 52	52 17	22 30	82 04	+ 1	+ 1	N	11 15	20

* a = number of turns of the micrometer screw in one minute of arc = 1.34. The value of one turn of the micrometer screw = $44''.650$.

78. Making the Observations.

In observing on a pair the finder circle is set for the mean of the two zenith distances, and the level is brought nearly to the center

of the tube. If the northerly star of the pair culminates first, the telescope is set on the north side of the meridian by means of the azimuth stop. When the star enters the field, the observer bisects it with the micrometer line. If a pair of lines is used, the star is centered in the space between the two. When the star is on the meridian, as shown by the chronometer reading, the bisection of the star is perfected; the latitude level is read immediately, and then the scale of the micrometer screw. As soon as these readings are recorded, the telescope is turned to the south side of the meridian and the bubble is brought to the center, if necessary, by moving the whole telescope. In leveling the bubble the tangent screw of the setting circle must not be disturbed in any case, because the accuracy of the method depends upon preserving a fixed relation between the direction of the zero micrometer reading and the axis of the latitude level. The slightest change in the angle between these two during the observations on a pair will render the observations worthless. When the southern star appears in the field the pointing and the readings are made exactly as for the northern star.

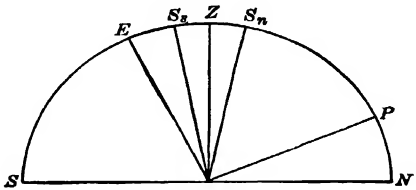

FIG. 80.

79. Formula for the Latitude.

The principle involved in this method may be seen in Fig. 80. The latitude, EZ, as derived from the southern star, is

$$EZ = ES_s + S_sZ,$$

or

$$\phi = \delta_s + \zeta_s,$$

and from the northern star it is

$$EZ = ES_n - ZS_n,$$

or

$$\phi = \delta_n - \zeta_n.$$

The mean of the two values of ϕ is

$$\phi = \frac{\delta_s + \delta_n}{2} + \frac{\zeta_s - \zeta_n}{2}. \qquad [29]$$

If we let

n_s and s_s = the level readings for the southerly star,
n_n and s_n = the level readings for the northerly star,
d = the angular value of one division of the level,
r_s and r_n = the refraction corrections,
M_s and M_n = the micrometer readings,
and R = the value of one turn of the micrometer,

then the latitude is determined by the equation

$$\phi = \frac{1}{2}(\delta_s + \delta_n) + \frac{1}{2}(M_s - M_n) \cdot R + \frac{d}{4}\{(n_s + n_n) - (s_s + s_n)\} + \frac{1}{2}(r_s - r_n). \qquad [30]$$

This formula applies when the zero of the level scale is in the center of the tube. If the zero is at the objective end of the tube, the level correction is

$$+\frac{d}{4}\{(n_s - n_n) + (s_s - s_n)\}.$$

If for any reason the observations are not made when the star is exactly on the meridian, another term must be added to the above formula; this will be of the form $+\frac{1}{2}(m_s + m_n)$ when m_s and m_n are the reductions of the measured zenith distances to the true zenith distances. (See *Special Publication No.* 14, p. 119.) For the application of least squares to the computation of latitude see Chauvenet, *Spherical and Practical Astronomy;* Hayford, *Geodetic Astronomy;* and *Coast and Geodetic Survey Special Publication No.* 14.

80. Calculation of the Declinations.

When the stars selected are not found in the Ephemeris, it will be necessary to calculate the apparent declinations for the date of the observation. Formulæ and tables for making these

reductions will be found in Part II of the Ephemeris. See also *Coast and Geodetic Survey Special Publication No.* 14, p. 116.

81. Correction for Variation of the Pole.

The observed latitude may be in error by several tenths of a second, owing to the fact that the observed value necessarily

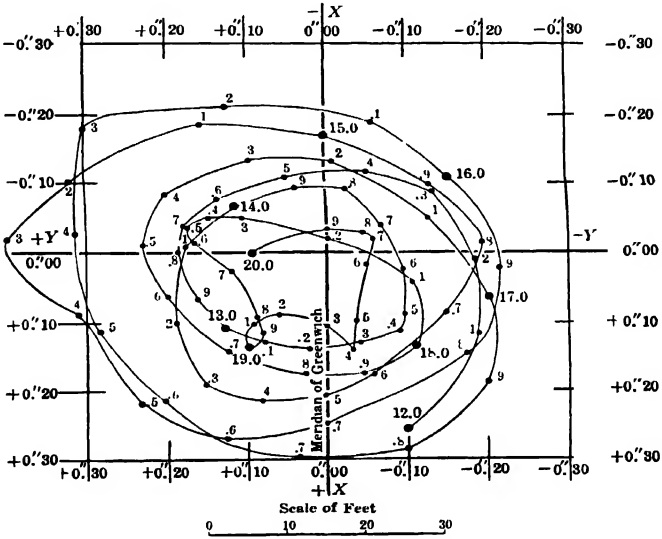

FIG. 81. Showing Changes in Position of North Pole between 1912 and 1920.

refers to the position of the pole at the date of the observation, whereas the fixed value of the latitude of a place is that referred to the mean position of the pole. Figure 81 shows the plotted positions of the pole for every 0.1 year during the period 1912.0 to 1920.0. The coördinates of the instantaneous pole and data for correcting observed values were formerly published by the International Geodetic Association. They are now (1929) published by Prof. H. Kimura in the Proceedings of the (Japanese) Imperial Academy.

82. Reduction of the Latitude to Sea-Level.

In order that all latitudes may refer to the same level surface, they are all reduced to their values at sea-level. If we suppose a lake surface, in the northern hemisphere, to be at a great height above sea-level, then it may be shown that the northern end of this lake surface is actually nearer to the surface of the sea than is the southern end of the lake surface. If we imagine a series of such surfaces at varying heights above sea-level, it is obvious that the *vertical* is a curved line, since it must at every point be normal to the level surface passing through that point. Evidently this curved line is concave toward the earth's rotation axis. To correct an observed latitude at elevation h to the corresponding latitude at sea-level, it is necessary to apply the correction

$$\Delta\phi = -0''.052\, h \sin 2\phi, \qquad [31]$$

where h is in thousands of feet. If h is expressed in meters, the formula becomes

$$\Delta\phi = -0.000171\, h \sin 2\phi. \qquad [32]$$

(See Art. 170, p. 325.) Values of this correction will be found in Table VII. Below is an example of the form of record and computation of latitude from *Special Publication No.* 14.

RECORD OF LATITUDE OBSERVATION.

[Station, St. Anne. Date, June 25, 1908. Chronometer, 2637. Observer, W. Bowie.]

No. of pair.	Star number Boss cat.	N or S.	Micrometer.		Level		Chronometer time of culmination.	Chronometer time of observation.	Meridian distance.	Remarks.
			Turns.	Div's.	North.	South.				
	4623	N	24	88.2	9.2	42.6				
					71.6	103.8		18 13 18		
11	4651	S	16	66.0	42.2	8.7	*	18 18 39	*	+16†
					103 2	71.0				
...	...	...					...			..

* These columns used only when star is observed off the meridian.

† This is the continuous sum, up to this pair, of the south minus the north micrometer turns.

LATITUDE COMPUTATION

Date.	Catalogue.		Micrometer.		Level			Merid-ian dis-tance.	Declination
	Star. No.	N or S.	Reading.	Diff. Z D.	N	S.	Diff.		
1908.				t. d.			d	S	° ′ ″
June 25	4623	N	24 88.2		09 2	42 6			64 21 59 5.
				−8 22.2	71.6	103 8			
	4651	S	16 66 0		42.2	08 7	−1 05		17 46 48 6.
					103 2	71 0			

Sum and half sum.	Corrections. Micrometer.	Level.	Refraction.	Meridian	Latitude
82 08 48 15 41 04 24 08	−3 03 56	−0 39	−0.06		41 01 20 07

Value of one division of latitude level: Upper — 1″ 600
Lower — 1 364
Mean — 1 482

Value of one turn of micrometer = 44″ 650

83. Accuracy of the Observed Latitude.

The latitude may be determined by this method with a probable error of from 0″.3 to 0″.4 from one pair of stars. The final value for the latitude of the station determined from as many pairs of stars as can be observed on one night may be found with an error of from 0″.05 to 0″.10 (or 5 to 10 feet). It is not considered advisable to observe the same pair of stars on several nights, as was formerly the practice, owing to the comparatively large errors in the declinations themselves. The present practice is to observe each pair but once and to observe such a number of pairs that the uncertainty of the final latitude is not greater than 0″.10.

In view of the fact that nearly every latitude is affected by a station error which may amount to several seconds, and that the real object of the observation is to determine this station error, it is better to determine a large number of latitudes with the degree of accuracy above mentioned than to attempt to diminish

the error of observation and occupy but a small number of stations. This results in the practice of occupying stations but one night, unless for some reason it is apparent that the required accuracy will not be reached without additional observations.

84. Determination of Azimuth.

When determining an azimuth for the purpose of orienting a triangulation system, the observer usually has a choice of several methods, all of them capable of yielding the required accuracy, for example, (1) measuring the angles between a circumpolar star and the triangulation lines by means of the direction instrument, (2) measuring from a triangulation station to a circumpolar star with the repeating instrument, or (3) measuring from a circumpolar star to an azimuth mark with the micrometer of a transit instrument. In all determinations of azimuth it is necessary to know the local time in order to compute the azimuth of the star. This must be found by special observations, unless, as is often the case, the longitude is being determined at the same time and the chronometer correction is already known. For the purpose of orienting the primary triangulation it is necessary to determine the azimuth with an error not exceeding $0''.50$. At Laplace stations (coincident triangulation, longitude, and azimuth stations), where the accumulated twist of the chain of triangles is to be determined, it is desirable to determine the azimuth within $0''.30$ or less. It is also desirable that the instrument station and the azimuth mark should both be triangulation stations. When horizontal angles are being measured at night, the azimuth observation is made a part of the same program by including pointings on a circumpolar star with the regular series of pointings on lights at the triangulation stations. An azimuth found by this method is more accurate than one determined by means of an auxiliary point and subsequently connected with the triangulation by means of a horizontal angle measured by daylight.

On account of the slow apparent motions of stars near the pole, nearly all accurate azimuth observations are made on close cir-

cumpolars, since errors of the latitude and the time have less effect on the result than for stars farther from the pole. The stars ordinarily used for azimuth observations are shown in Fig. 82.

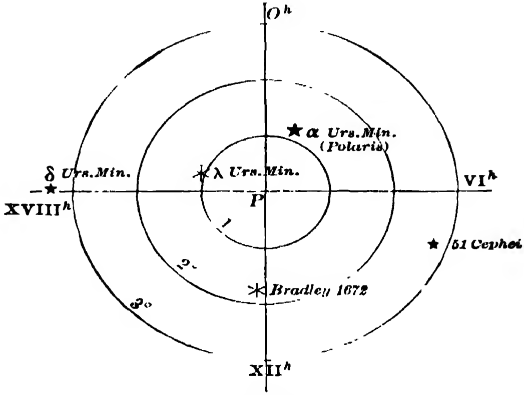

FIG. 82. Circumpolar Stars.

85. Formula for Azimuth.

In general all these methods consist in calculating the azimuth of the star at the instant of observation and combining this azimuth with the measured horizontal angle from the star to the station. The azimuth of a circumpolar star is found by the formula

$$\tan Z = -\frac{\sin t}{\cos \phi \tan \delta - \sin \phi \cos t} \qquad [33]$$

where Z is the azimuth measured from the north toward the east, and t is the hour angle.

If Equa. [33] be divided by $\cos \phi \tan \delta$, then

$$\tan Z = \frac{\cot \delta \sec \phi \sin t}{1 - \cot \delta \tan \phi \cos t}$$

$$= -\cot \delta \sec \phi \sin t \left(\frac{1}{1-a}\right). \qquad [34]$$

If values of $\frac{1}{1-a}$ are tabulated,* this formula will be found more convenient than Equa. [33].

86. Curvature Correction.

In computing the azimuth of the star it would evidently be inconvenient to apply the formula to each separate pointing on the star, on account of the large amount of computation. It is simpler and sufficiently accurate to calculate the azimuth of the star at the *mean* of the observed times of pointing, and then to correct the computed azimuth for the small difference between this azimuth and the mean of all the azimuths. The correction for this difference is

$$\text{Curvature Correction} = -\tan Z \frac{1}{n} \sum \frac{2 \sin^2 \frac{\tau}{2}}{\sin 1''}, \qquad [35]$$

in which n = the number of pointings
and τ = the interval of time between the observed time and the mean.

The sign of the correction is such that it always decreases the angle between the star and the pole. For the derivation of this formula see Hayford's *Geodetic Astronomy*, p. 213. The correction may also be written in the form $-\tan Z\,[6.73672]\,\frac{1}{n}\sum \tau^2$. (See Doolittle's *Practical Astronomy*, p. 537.)

87. Correction for Diurnal Aberration.

On account of the motion of the observer, due to the earth's rotation, the star is apparently displaced toward the east. The correction to the computed azimuth for the effect of this apparent displacement is given by the expression

$$\text{Corr. for Aberra.} = 0''.32 \frac{\cos Z \cos \phi}{\cos h}. \qquad [36]$$

This correction is always positive for an azimuth counted clock-

* For a table of values of log $\frac{1}{1-a}$ see *Special Pub. No.* 14.

wise. For the derivation of this formula see Doolittle's *Practical Astronomy*, p. 530.

88. Level Correction.

If the horizontal axis is not level when a pointing is made on the star, the observed direction must be corrected by the following quantity:

$$\text{Lev. Corr.} = \frac{d}{4}[(w + w') - (e + e')] \tan h. \qquad [37]$$

For proof of this formula see pp. 83 and 126. If the level is graduated from one end to the other

$$\text{Lev. Corr.} = \frac{d}{4}[(w - w') + (e - e')] \tan h, \qquad [38]$$

where w and e are read before, and w' and e' are read after, the reversal of the striding level. If the azimuth mark is not near the horizon, it is necessary to apply a similar correction to the observed direction of the mark. The correction is to be added algebraically to readings which increase in a clockwise direction.

89. The Direction Method.

In observing for azimuth by this method the measurements are carried out very nearly as they are for triangulation, except that the chronometer is read whenever the star is sighted and that level readings are taken to determine the inclination of the axis and altitude readings of the star are taken.

The observations should be taken in several positions of the circle. In each position of the circle the observations will include (1) a pointing on the mark, with readings of the microscope, (2) a pointing on the star, with reading of chronometer, striding level and microscopes, (3), after reversal of the telescope, a pointing on the star with readings of the chronometer, striding level and microscopes, and (4) a pointing on the mark with readings of the microscopes. If the different steps are taken in the following order it will be found to economize time and yet give the bubble of the striding level time to settle and also give the recorder ample time to read the chronometer and record the

time. After completing the pointing and reading on the mark sight the telescope approximately on the star, and call for the recorder to be ready. Place the striding level in position, point accurately on the star and call "tip" to the recorder when the pointing is made. Read the two ends of the bubble, but do not give them to the recorder until his time record has been made. Reverse the striding level, and prepare to read the first micrometer. As soon as the recorder calls "ready" the first level readings are given to him, and then the micrometer readings, in their proper order. The second (reversed) readings of the striding level are taken and called off to the recorder. This completes half the set for this position of the circle. Remove the striding level, reverse the telescope, bring the star to the center of the field, call for the recorder to be ready, place the striding level in position, then perfect the pointing on the star, etc. Complete the readings exactly as for the first half. Finally make a second pointing on the mark. The altitude of the star should be read to the nearest minute after each pointing, or at least twice during the observations.

If the azimuth mark is far above or below the horizon, level readings should be taken when the mark is sighted.

If the azimuth observation is being taken as a part of the program of triangulation at night the above directions should be modified accordingly. The signals should be sighted in order around the horizon, the star being sighted in its turn; after reversing the telescope the signals should be sighted in the reverse order, the star being sighted again in its turn. The details of the sight on the star would be the same as already described.

On pp. 157 and 158 is an example of the form of record and computation of an azimuth by the method of directions.

HORIZONTAL DIRECTIONS

[Station, Sears, Tex. (Triangulation Station). Observer, W. Bowie. Instrument, Theodolite 168. Date, Dec. 22, 1908.]

Position.	Objects observed.	Time.	Tel. D or R.	Mic.	Backward.			For-ward.	Mean.	Mean D and R.	Direc-tion.	Remarks.
		h m			°	′	″	″			″	
1	Morrison	8 19	D	*A*	0	0	35	35				1 division of the
				B			41	41				striding level
				C			36	34	37 0			= 4″ 194
			R	*A*	180	00	36	35				
				B			32	31				
				C			35	34	33 8	35 4	00 0	
	Buzzard		D	*A*	53	30	43	42				
				B			41	42				
				C			34	33	39 2			
			R	*A*	233	30	39	37				
				B			34	32				
				C			38	38	36 3	37 8	02 4	
	Allen		D	*A*	170	14	61	62				
				B			57	55				
				C			61	59	59 2			
			R	*A*	350	14	50	49				
				B			63	60				
				C			53	53	54 7	57 0	21 6	
	Polaris.		D	*A*	252	01	54	53				W E
	h m s			*B*			54	53				9 3 28 0
	1 48 35 5			*C*			51	51	52 7			27 7 9 1
	1 51 06 0											18 4 − 0 5 18 9
	1 49 50 8											
			R	*A*	72	01	09	09				24 9 6 3
				B			02	01				13 0 31 7
				C			10	08	06 5	29 6		
												11 9 −13 5 25.4
												− 7.0

COMPUTATION OF AZIMUTH, DIRECTION METHOD.

[Station, Sears, Tex. Chronometer, sidereal 1769. $\phi = 32° 33' 31''$
Instrument, theodolite 168. Observer, W. Bowie.]

Date, 1908, position	Dec 22, 1	2	3	4
Chronometer reading	1 49 50 8	2 01 33 0	2 16 31 0	2 43 28 8
Chronometer correction	— 4 37 5	— 4 37 5	— 4 37 4	— 4 37 3
Sidereal time	1 45 13 3	1 56 55 5	2 11 53 6	2 38 51 5
α of Polaris	1 26 41 9	1 26 41 9	1 26 41 8	1 26 41 8
t of Polaris (time)	0 18 31 4	0 30 13 6	0 45 11 8	1 12 09 7
t of Polaris (arc)	4° 37′ 51″ 0	7° 33′ 24″ 0	11° 17′ 57″ 0	18° 02′ 25″ 5
δ of Polaris	88 49 27 4			
log cot δ	8 31224	8 31224	8 31224	8 31224
log tan ϕ	9 80517	9 80517	9 80517	9 80517
log cos t	9 99858	9 99621	9 99150	9 97811
log a (to five places)	8 11599	8 11362	8 10891	8 09552
log cot δ	8 312243	8 312243	8 312243	8 312243
log sec ϕ	0 074254	0 074254	0 074254	0 074254
log sin t	8 907064	9 118948	9 292105	9 490924
$\log \frac{1}{1-a}$	0 005710	0 005679	0 005618	0 005445
log (−tan A) (to 6 places)	7 299271	7 511124	7 684220	7 882866
A = Azimuth of Polaris, from north*	0 06 50 8	0 11 09 2	0 16 36 9	0 26 15 0
Difference in time between D and R.	m s 2 30	m s 2 00	m s 3 18	m s 1 38
Curvature correction	0	0	0	0
	° ′ ″	° ′ ″	° ′ ″	° ′ ″
Altitude of Polaris = h	33 46	33 46	33 46	33 46
$\frac{d}{4}$ tan h = level factor	0 701	0 701	0 701	0 701
Inclination †	−7 0	−7.2	−7 0	−1 8
Level correction	−4 9	−5 0	−4 9	−1 3
Circle reads on Polaris	252 01 29 6	86 58 11 2	281 54 27 0	116 45 48 6
Corrected reading on Polaris	252 01 24 7	86 58 06 2	281 54 22 1	116 45 47 3
Circle reads on mark	170 14 57 0	5 15 58 2	200 17 42 4	35 18 45 4
Difference, mark — Polaris	278 13 32 3	278 17 52 0	278 23 20 3	278 32 58 1
Corrected azimuth of Polaris, from north *	0 06 50 8	0 11 09 2	0 16 36 9	0 26 15 0
	180 00 00 0	180 00 00 0	180 00 00 0	180 00 00 0
Azimuth of Allen (Clockwise from South)	98 06 41 5	98 06 42 8	98 06 43 4	98 06 43 1

To the mean result from the above computation must be applied corrections for diurnal aberration and eccentricity (if any) of Mark.

Carry times and angles to tenths of seconds only.

* Minus, if west of north.

† The values shown in this line are actually four times the inclination of the horizontal axis in terms of level divisions.

90. Method of Repetition.

In observing by the repetition method the program given on p. 89 is followed, with the addition of readings of the chronometer and the stride level, taken when the telescope is pointing at the star. The altitude of the star should be measured, if possible, but may be computed from the known time if necessary. The verniers are read only at the beginning and end of a half-set, as when measuring the angles of a triangulation.

Following is an example of the form of record and computation of an azimuth by the method of repetition.

RECORD — AZIMUTH BY REPETITIONS.

[Station, Kahatchee △. State, Alabama. Date, June 6, 1898. Observer, O. B. F. Instrument, 10-inch Gambey No. 63. Star, Polaris.]

[One division striding level = 2″.67.]

Objects	Chr. time on star.	Pos. of tel.	Repetitions.	Level readings. W.	Level readings. E.	Circle readings. °	′	A ″	B ″	Mean	Angle.
	h m s										° ′ ″
Mark ..		D	0			178	03	22 5	20	21 2	
Star .	14 46 30		1	4 5	10 7						
				9 2	5 9						
	49 08		2								
	52 51	D	3	9 6	5 6						
				5 2	12 0						
	56 10	R	4	11 3	4 0						
				7 8	7 4						
Set No. 5..	14 59 12		5								
	15 01 55	R	6	8.7	6 6	100	16	20	20	20	72 57 50.2
				11 9	3 4						
	14 54 17 7			68 2	53 6						
				+	14 6						
Star . .	15 04 44	R	1	11 9	3 4						
				8.5	6 8						
	07 18		2								
	09 54	R	3	7 9	7 3						
				11 2	4 1						
Set No. 6 .	14 15	D	4	9 0	6.1						
				5 9	9 6						
	16 14		5								
	15 18 24		6	5 9	9 6						
				9 1	6 2						
Mark		D				177	27	00	00	00	72 51 46.7
	15 11 48 2			69 4	53 1						
					+16 3						

COMPUTATION — AZIMUTH BY REPETITIONS

[Kahatchee, Ala. $\phi = 33° 13' 40''.33$.]

Date, 1898, set	June 6 5	June 6 6
Chronometer reading	14 54 17.7	15 11 48.2
Chronometer correction	−31.1	−31.1
Sidereal time	14 53 46.6	15 11 17.1
α of Polaris	1 21 20.3	1 21 20.3
t of Polaris (time)	13 32 26.3	13 49 56.8
t of Polaris (arc)	203° 06′ 34″.5	207° 29′ 12″.0
δ of Polaris	88 45 46.9	
log cot δ	8.33430	8.33430
log tan ϕ	9.81629	9.81629
log cos t	9.96367n	9.94798n
log a (to five places)	8.11426n	8.09857n
log cot δ	8.334305	8.334305
log sec ϕ	0.077535	0.077535
log sin t	9.593830n	9.664211n
$\log \frac{1}{1-a}$	9.994387	9.994584
log ($-\tan A$) (to 6 places)	8.000057n	8.070635n
A = Azimuth of Polaris, from north *	0° 34′ 22″.8	0° 40′ 26″.8
	m s ″	m s ″
τ and $\frac{2 \sin^2 \frac{1}{2}\tau}{\sin 1''}$	7 47.7 119.3	7 04.2 98.1
	5 09.7 52.3	4 30.2 39.8
	1 26.7 4.1	1 54.2 7.1
	1 52.3 6.9	2 26.8 11.8
	4 54.3 47.2	4 25.8 38.5
	7 37.3 114.0	6 35.8 85.4
Sum	343.8	280.7
Mean	57.3	46.8
$\log \frac{1}{n} \sum \frac{2 \sin^2 \frac{1}{2}\tau}{\sin 1''}$	1.758	1.670
log (curvature corr.)	9.758	9.741
Curvature correction	−0.6	−0.6
Altitude of Polaris = h	32° 07′	
$\frac{d}{4}$ tan h = level factor	0.419	0.419
Inclination	+3.6	+4.1
Level correction	−1″.5	−1″.7
Angle, star — mark	72 57 50.2	72 51 46.7
Corrected angle	72 57 48.7	72 51 45.0
Corrected azimuth of star *	0 34 22.2	0 40 26.2
Azimuth of mark E of N	73 32 10.9	73 32 11.2
	180 00 00.0	180 00 00.0
Azimuth of mark (Clockwise from south)	253 32 10.9	253 32 11.2

To the mean result from the above computation must be applied corrections for diurnal aberration and eccentricity (if any) of Mark. Carry times and angles to tenths of seconds only.

*** Minus if west of north.**

91. Micrometric Method.

In employing this method it is necessary to place a mark nearly in the same vertical plane with the star at the time of the observation. For greatest accuracy, as well as for convenience, the star should be observed when near its greatest elongation. Near culmination the star's motion will carry it beyond the range of the micrometer in a comparatively short time. The small difference in azimuth between the star and the mark is to be measured with the micrometer in the eye-piece of a transit instrument. The instrument is clamped in azimuth, and the readings are taken in the following order: take five pointings on the mark; point toward the star and place the stride level in position; take three pointings on the star with their corresponding chronometer times; read and reverse the stride level; take two more pointings on the star, noting the times; read the stride level; reverse the horizontal axis of the instrument in the bearings, point the telescope at the star, and place the level in position; take three pointings on the star, with chronometer times; read the level and reverse it; take two more pointings on the star and the times; read the level; finally, take five pointings on the mark. Three such sets will be found to require from thirty to fifty minutes' time. Either the altitude or the zenith distance of the star should be read twice during the set, in order that an altitude for use in calculating the azimuth may be interpolated.

The angle given by the micrometer readings is in the plane of the line of collimation and the horizontal axis. To reduce this angle to the horizontal plane, multiply it by the secant of the altitude. Each half-set may be reduced separately. The altitude for the middle of each half-set may be used for reducing to horizontal. The value of one turn of the micrometer screw may be found by observing a circumpolar star near culmination, or, better still, by measuring a small angle by means of a theodolite and then measuring this angle with the micrometer.

Following is an example of record and computation.

RECORD AND COMPUTATION — AZIMUTH BY MICROMETRIC METHOD

[Station No. 10, Mexican Boundary. Date, Oct. 13, 1892. Observer, J. F. H. Instrument, Fauth Repeating Theodolite, No. 725 (10 in.). Star, Polaris near eastern elongation.]

Circle.	Level readings. W	Level readings. E	Chronometer time.	τ.	$\frac{2 \sin^2 \frac{1}{2}\tau}{\sin 1''}$	Micrometer readings — On star.	Micrometer readings — On mark.	
			h m s	m s				
E	8 0	9 9	9 06 38 0	3 58 6	31 05	18^t 379	18^t 310	$\lambda = 2^h\ 12^m$ W of
	10	7 3	07 32 0	3 04 6	18 59	0 388	0 315	Washington
								$\phi = 31°\ 19'\ 35''$
	+18 0	−17 2	08 05 5	2 31 1	12 45	0.400	0 315	1 div. of level
	+0 8		09 13 0	1 23 6	3 82	0 424	0 311	= 3'' 68
E			09 48 0	0 48 6	1 29	0 430	0 316	1 turn of mic.
								= 123'' 73
						18 4042	18 3134	Means
W	9 0	9 0	9 12 01 8	1 25 2	3 96	18 100	18 290	
	7 0	10 9	12 24 7	1 48 1	6 37	0 100	0 275	
	+16 0	−19 9	12 48 3	2 11 7	9 46	0 090	0 279	
	−3 9		13 36 3	2 59 7	17 61	0 086	0 281	
W	Mean	$1^d.55$	13 58 1	3 21 5	22 14	0 080	0 279	
			9 10 36 6		12 67	18 0912	18 2808	Means

ζ of star at middle of first half of set = 58° 48'. cosec ζ = 1 1691. cot 58° 47' = 0 606.
ζ of star at middle of second half of set = 58° 46'. cosec ζ = 1.1695.
$\alpha = 1^h\ 20^m\ 07^s.4$. $\delta = 88°\ 44'\ 10''.4$.

Collimation axis reads ½ (18.3134 + 18.2808)* = 18^t 2971

* In this instrument increased readings of the micrometer correspond to a movement of the line of sight toward the east when the vertical circle is to the east, and toward the west when the vertical circle is to the west.

Mark east of collimation axis 18.3134 − 18.2971	= 0 0163	= 02''.02	
Circle E., star E of collimation axis (18.4042 − 18.2971) (1.1691)	= 0 .1252		
Circle W., star E of collimation axis (18.2971 − 18.0912) (1.1695)	= 0 2408		
Mean, star E of collimation axis	= 0 1830	= 22 64	
Mark west of star		= 20 62	
Level correction (1.55) (0.92) (0.606)		= −0 86	
Mark west of star, corrected		= 19 76	

Mean chronometer time of observation	= $21^h\ 10^m\ 36^s\ 6$
Chronometer correction	= − 2 11 28 .2
Sidereal time	= 18 59 08 4
α	= 1 20 07 4
Hour-angle, t, in time	17 39 01 .0
" in arc	264° 45' 15''.0

log cot δ	= 8 34362
log tan ϕ	= 9 78436
log cos t	= 8 96108 n
log a	= 7.08906 n
log cot δ	= 8 343618
log sec ϕ	= 0 068431
log sin t	= 9 998177 n
log $\frac{1}{1-a}$	= 9 999467
log (−tan A)	= 8 400693 n
A	= +1° 28' 16'' 91
log. 12.67	= 1 10278
log. curvature corr.	= 9 51247

Curvature corr.	= −0.33
Diur. Aber. corr.	= +0 32
Mean azimuth of star	= +1° 28' 16''.90
Mark west of star	19 76
Azimuth of mark, E of N	= +1° 27' 57'' 14

92. Reduction to Sea-Level.

If the azimuth mark is at a high elevation, the computed azimuth must be reduced to its value at the point where the vertical through the mark intersects the sea-level. This correction in seconds is

$$+\frac{e^2h}{2\,a\sin 1''}\cdot\cos^2\phi\sin 2\,\alpha, \qquad [39]$$

in which h is the elevation, ϕ is the latitude, α is the azimuth, and e and a are for the Clarke Spheroid of 1866 (see Art. 102, p. 182). If h is expressed in meters, this becomes

$$+0''.000109\,h\cos^2\phi\sin 2\,\alpha. \qquad [40]$$

$$(\log \text{ of } 0.000109 = 6.0392 - 10.)$$

If the mark is either northeast or southwest of the observing station the observed azimuth must be increased to obtain the correct azimuth; if the mark is northwest or southeast, the observed azimuth must be decreased.

Reduction to Mean Position of the Pole.

The observed azimuth must be reduced to its value corresponding to the mean position of the pole. In latitude 50° (northern United States) this correction may be as great as half a second (see p. 149).

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PROBLEMS

Problem 1. What should be the linear distance between the vertical threads of a transit having a 30-inch focus in order to give $2^s.5$ intervals of time between threads for an equatorial star?

Problem 2. The following readings were taken to determine the pivot inequality of a transit. Clamp east, level direct, $w = 43.5$, $e = 34.0$; level reversed, $w = 36.7$, $e = 41.0$. Clamp west, level direct, $w = 39.1$ $e = 37.0$; level reversed, $w = 34.2$,

$e = 41.8$ The value of one division of the level is $0''.75$. This level has the zero at the center and is numbered both ways. Find the pivot inequality.

If a star is observed with the transit in the position clamp east what is the level correction to the observed time of transit if $\delta = +30°$ and $\phi = +40°$?

Problem 3. If the collimation axis of a transit has a true bearing of S 0° 00′ 15″ E what is the correction to the observed time of transit of a star if $\delta = +39°$ and $\phi = +30°$?

Problem 4. If a latitude is found to be 36° 49′ 50″.261 at an altitude of 6250 feet what will this latitude be when reduced to sea-level?

Problem 5. Compute the latitude from the following zenith telescope observations.

Star No. 2125, south; chr. time $13^h\ 37^m$; micrometer $16^t.063$; level, n 83.0, s 30.0. Star No. 2141, north; chr. time, $13^h\ 43^m$; micrometer, $13^t.504$; level n 31.0, s 83.5. Eye-piece on side toward micrometer head; level zero on side opposite to eye-piece. Declination of 2125, 28° 34′ 09″.80; declination of 2141, 39° 00′ 08 80. One division of latitude level = 1″.00. One turn of micrometer = 155″.80. In this case the micrometer readings decrease as the zenith distances increase; the sign is therefore the opposite of that given in Equa. [30] p. 148.

CHAPTER V

PROPERTIES OF THE SPHEROID

93. Mathematical Figure of the Earth.

In calculating the positions of survey points on the earth, it is necessary to consider these points as lying upon some mathematical surface, like the sphere or the ellipsoid, taken to represent the figure of the earth. This is accomplished by projecting the position of the station vertically downward onto the surface in question. The actual shape of the earth's surface is quite irregular and, from the nature of the problem, can only be determined approximately. But even if it could be found exactly, it would not be adapted to the purpose of computation. For this reason it is necessary to select some figure, the use of which will simplify the computation, but which will nowhere depart from the true figure by an amount sufficient to produce serious errors in the results. The figure generally adopted is the *oblate spheroid* or *ellipsoid of revolution*. Such a figure is generated by rotating an ellipse about its shorter axis. This surface approaches much nearer the actual figure of the earth than does the sphere, but perhaps not quite so near as an ellipsoid of three unequal dimensions. The latter, however, would be an inconvenient figure to use, and the gain in accuracy would be very slight.

The oblate spheroid is an ellipsoidal surface with two of its axes equal, but with the third axis, about which the figure rotates, shorter than the other two. All plane sections of such a surface are ellipses, except those cut by planes perpendicular to the rotation axis. Sections through the rotation, or polar, axis are ellipses whose major axes are the equatorial diameter, and whose minor axes are the polar diameter, of the spheroid. The nature

of this surface will be understood best if we investigate first the properties of the ellipse which generates the spheroid.

94. Properties of the Ellipse.

In Fig. 83, PP' is the polar axis of the spheroid, and EE' is any one of the equatorial diameters. F is one focus of the ellipse. At M, any point on the curve, the line MA is drawn tangent to the ellipse; MH is perpendicular to the tangent, that is, normal

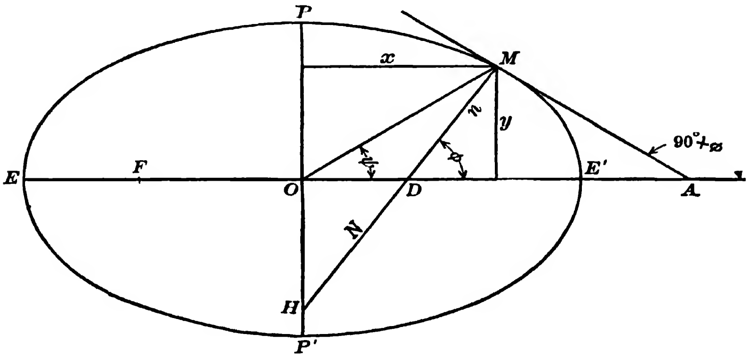

FIG. 83.

to the curve. MH is the direction that the plumb line at M is supposed to assume unless deflected by local causes, such as variations in density. The distance MH $(=N)$, terminating in the minor axis, is called the *normal*. MD $(=n)$ is the normal terminating in the major axis. The angle made by the normal with OE', that is, with the plane of the earth's equator, is the *geodetic latitude* (ϕ).* The angle made by MO with OE' is the *geocentric latitude* (ψ).

Another angle which is of importance in the geometry of the ellipse is the *eccentric angle*, or *reduced latitude*, θ. It is the angle $E'Om$, Fig. 84, in which M is any point on the ellipse, MN is

* The *astronomical latitude* is the angle made by the actual direction of gravity (plumb line) with the plane of the equator.

perpendicular to OE', and m is the point where this perpendicular cuts the circle whose center is O and radius OE'.

The equation of the ellipse whose major and minor semiaxes are a and b, referred to its own axes as coördinate axes, is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

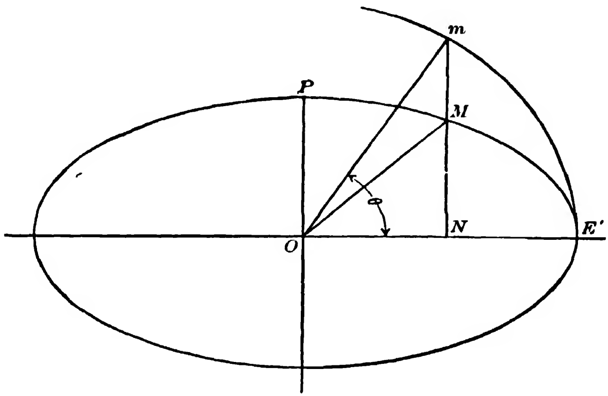

FIG. 84.

To determine the coördinates of any point M (Fig. 83), in terms of the latitude, differentiate this equation and the result is

$$\frac{y}{x} = -\frac{b^2}{a^2}\frac{dx}{dy}. \tag{1}$$

Since the tangent line to an ellipse makes an angle with the axis of X whose tangent is $\dfrac{dy}{dx}$,

or
$$\tan(90° + \phi) = \frac{dy}{dx},$$

$$\therefore \quad \tan\phi = -\frac{dx}{dy}.$$

The eccentricity e is the distance from the focus to the center

divided by a, that is $\frac{OF}{OE}$. From the triangle OFP it will be seen that

$$e^2 = \frac{a^2 - b^2}{a^2},$$

or

$$\frac{b^2}{a^2} = 1 - e^2.$$

Therefore (1) may be written

$$\frac{y}{x} = (1 - e^2) \tan \phi. \tag{2}$$

From the equation of the ellipse,

$$x^2 + \frac{y^2}{1 - e^2} = a^2. \tag{3}$$

Squaring (2) and substituting in the result the value of y^2 from (3), we obtain*

$$x = \frac{a \cos \phi}{\sqrt{1 - e^2 \sin^2 \phi}} \tag{41}$$

and

$$y = \frac{a(1 - e^2) \sin \phi}{\sqrt{1 - e^2 \sin^2 \phi}}. \tag{42}$$

95. Radius of Curvature of the Meridian.

To find the radius of curvature of the meridian (R_m), apply the general formula

$$R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

From (1)

$$\frac{dy}{dx} = -\frac{x}{y}\frac{b^2}{a^2}.$$

* The relation $1 + \tan^2 \phi = \sec^2 \phi$ is used in this transformation.

Differentiating this equation, we have

$$\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\left(\frac{y - x\dfrac{dy}{dx}}{y^2}\right)$$
$$= -\frac{b^2}{a^2y^2}\left(y + \frac{x^2}{y}\cdot\frac{b^2}{a^2}\right)$$
$$= -\frac{b^4}{a^2y^3}.$$

Therefore
$$R_m = -\frac{\left[1 + \dfrac{x^2}{y^2}\dfrac{b^4}{a^4}\right]^{\frac{3}{2}}}{\dfrac{b^4}{a^2y^3}}$$
$$= -\frac{[a^4y^2 + b^4x^2]^{\frac{3}{2}}}{a^4b^4}$$
$$= -\frac{\left[\dfrac{a^6(1 - e^2)^2\sin^2\phi}{1 - e^2\sin^2\phi} + \dfrac{b^4a^2\cos^2\phi}{1 - e^2\sin^2\phi}\right]^{\frac{3}{2}}}{a^4b^4}.$$

Then, since $b^2 = a^2(1 - e^2)$,

$$R_m = -\frac{a(1 - e^2)}{(1 - e^2\sin^2\phi)^{\frac{3}{2}}}.* \qquad [43]$$

Values of log R_m will be found in Table X. For latitudes 0°, 45°, and 90° these are as follows:

Lat	log R_m	R_m	Difference	
0°	6 8017489	6 335 033^m		
			32 298^m	20.07 miles
45°	6 8039574	6 367 331		
			32 570^m	20.24 "
90°	6.8061733	6 399 901		

96. Radius of Curvature in the Prime Vertical.

The radius of curvature of the surface of the spheroid in a plane through the normal and at right angles to the meridian may be proved to be equal to the length of the normal (N)

* The negative sign indicates only the direction of bending. It is customary to regard the value of R_m as positive.

terminating in the minor axis. If a section be taken through the center of the ellipsoid in a plane at right angles to the meridian its semiaxes will be a and $OM = x \sec \psi = \dfrac{a \cos \phi \sec \psi}{\sqrt{1 - e^2 \sin^2 \phi}}$ (see Fig. 83). The radius of curvature at the end of the minor axis of any ellipse is $a^2 \div b$. For the central section this gives $\rho = \dfrac{a\sqrt{1 - e^2 \sin^2 \phi}}{\cos \phi \sec \psi}$ for the point M. From Meunier's Theorem we know that the radius of curvature of the normal section (ρ') equals the radius of curvature of the inclined section (ρ) divided by the cosine of the angle between the two planes. From Fig. 83 this angle is seen to be $\phi - \psi$. Therefore

$$\begin{aligned}
\rho' &= \frac{a\sqrt{1 - e^2 \sin^2 \phi}}{\cos \phi \sec \psi \cos (\phi - \psi)} \\
&= \frac{a\sqrt{1 - e^2 \sin^2 \phi}}{\cos^2 \phi + \sin \phi \cos \phi \tan \psi} \\
&= \frac{a\sqrt{1 - e^2 \sin^2 \phi}}{\cos^2 \phi + \sin \phi \cos \phi\, (1 - e^2) \tan \phi}.
\end{aligned}$$

By Equa. [51]

$$\begin{aligned}
&\frac{a\sqrt{1 - e^2 \sin^2 \phi}}{\cos^2 \phi + \sin^2 \phi - e^2 \sin^2 \phi} \\
&= \frac{a}{\sqrt{1 - e^2 \sin^2 \phi}} \\
&= N \qquad \text{(by Equa. [44])}
\end{aligned}$$

Or, ρ' may be found directly from Equa. [46], since in this case $\rho = R_p = N \cos \phi$, and the angle between the sections is ϕ. Therefore $\rho' = N$.

To show this geometrically, let A and B in Fig. 85 be two points on the same parallel of latitude. The normals to the surface at A and B always intersect at H on the minor axis. Let C be a point on the prime vertical section through A, and also on the meridian of B. The normals to this plane curve at points A and C

intersect each other at some point K, above H. Therefore, K is approximately the center of curvature of the arc AC. Observe that CK is not normal to the surface. When the meridian PBC is taken nearer to A, points A and C approach each other, the intersection of the normals to the plane curve AC approaches the true center of curvature, and the length CK approaches the true radius of curvature. But the nearer C approaches A, the nearer it approaches B and the nearer CK becomes normal to the surface. Hence CK must ultimately coincide with AH; that is, H is the point toward which the center of curvature is moving and the normal N is the radius of curvature of the prime vertical

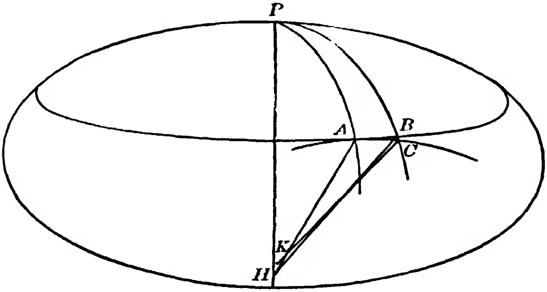

FIG. 85.

section at A. Note that BH and CK do not really intersect in space, although they appear to on the diagram.

From Fig. 83 it is evident that

$$N = \frac{x}{\cos \phi}$$

$$\sqrt{1 - e^2 \sin^2 \phi} \qquad [44]$$

Values of log N will be found in Table X.

Notice that the difference between N and R_m is greatest at the equator (about 26 miles); at the pole N and R_m become identical.

The normal terminating in the major axis is

$$n = \frac{y}{\sin\phi} = \frac{a(1 - e^2)}{\sqrt{1 - e^2 \sin^2\phi}} = N(1 - e^2). \qquad [45]$$

The radius of the parallel of latitude ($=x$) is given by

$$R_p = N\cos\phi. \qquad [46]$$

97. Radius of Curvature of Normal Section in any Azimuth. Having found the radii of curvature of the two principal sections, it now remains to find a general expression for the radius of curvature in any azimuth, and it will be shown that this may be expressed in terms of the two radii already found.

The equation of the spheroid is

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{a^2} + \frac{z_1^2}{b^2} = 1,$$

or

$$b^2x_1^2 + b^2y_1^2 + a^2z_1^2 = a^2b^2. \qquad (a)$$

In Fig. 86 the Z_1-axis coincides with the polar axis of the spheroid. If M be any point on the meridian Z_1M, and MY any section cut by a plane through MH (the normal) making an angle α with the meridian, then the equation of the spheroid may be transformed so as to refer to the origin C, the new Z axis CM, and a new Y axis at right angles to CM. Let the coördinates of any point P be x_1, y_1, z_1, and let the new coördinates be x, y, z. Then, from Fig. 86, the relation of the new coördinates to the old is given by

$$\begin{aligned} x_1 &= OG = OC + x + z\cos\phi + y\cos\alpha\sin\phi \\ &= Ne^2\cos\phi + x + z\cos\phi + y\cos\alpha\sin\phi, \\ y_1 &= y\sin\alpha, \\ z_1 &= z\sin\phi - y\cos\alpha\cos\phi. \end{aligned}$$

Substituting these values in (a),

$$b^2(Ne^2\cos\phi + x + z\cos\phi + y\cos\alpha\sin\phi)^2 + b^2y^2\sin^2\alpha + a^2(z\sin\phi - y\cos\alpha\cos\phi)^2 = a^2b^2,$$

which is the equation of the spheroid referred to the new axes. If x is made equal to zero, then P will be on the curve MY, and

the equation becomes the equation of this plane section, that is,

$$b^2(Ne^2 \cos\phi + z\cos\phi + y\cos\alpha\sin\phi)^2 + b^2y^2\sin^2\alpha$$
$$+a^2(z\sin\phi - y\cos\alpha\cos\phi)^2 = a^2b^2,$$

the equation of the ellipse MY.

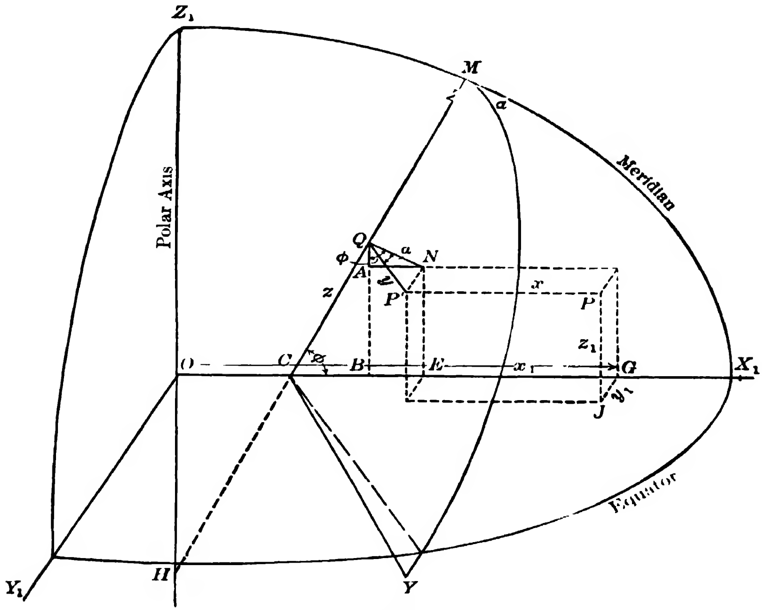

FIG. 86.

To determine the radius of curvature at M it is necessary to find $\frac{dz}{dy}$ and $\frac{d^2z}{dy^2}$ and to substitute these values in the general formula for radius of curvature.

Expanding the last equation, collecting terms, and dividing through by a^2,

$$y^2[1 - e^2(1 - \cos^2\alpha\cos^2\phi)] + z^2(1 - e^2\cos^2\phi)$$
$$-yz(2e^2\cos\alpha\sin\phi\cos\phi) + 2y(1 - e^2)\cdot N\cdot e^2\cos\alpha\sin\phi\cos\phi$$
$$+2ze^2(1 - e^2)\cdot N\cdot\cos^2\phi = (1 - e^2)(a^2 - Ne^4\cos^2\phi),$$

or, in abbreviated form,

$$y^2A + z^2B - yzC + yD + zE = F.$$

Differentiating this equation, y being taken as the independent variable,

$$2\,yA + 2\,z\frac{dz}{dy}B - Cy\frac{dz}{dy} - Cz + D + E\frac{dz}{dy} = 0.$$

Differentiating again,

$$2\,A + 2\,B\left(z\frac{d^2z}{dy^2} + \left(\frac{dz}{dy}\right)^2\right) - C\left(y\frac{d^2z}{dy^2} + \frac{dz}{dy}\right) - C\frac{dz}{dy} + E\frac{d^2z}{dy^2} = 0,$$

$$\frac{d^2z}{dy^2}(2\,Bz - Cy + E) = -\left(2\,A + 2\,B\left(\frac{dz}{dy}\right)^2 - 2\,C\frac{dz}{dy}\right),$$

$$\frac{d^2z}{dy^2} = -\frac{2\,A + 2\,B\left(\frac{dz}{dy}\right)^2 - 2\,C\frac{dz}{dy}}{2\,Bz - Cy + E}.$$

For point M, $y = 0$ and $z = n = N\,(1 - e^2)$. Therefore

$$\frac{dz}{dy} = -\frac{N\,(1-e^2)(2\,e^2\cos\alpha\sin\phi\cos\phi) - 2\,(1-e^2)\,Ne^2\cos\alpha\sin\phi\cos\phi}{2\,Bz - Cy + E} = 0,$$

and

$$\frac{d^2z}{dy^2} = -\frac{2\,[1 - e^2\,(1 - \cos^2\alpha\cos^2\phi)]}{2\,N(1 - e^2)\,(1 - e^2\cos^2\phi) + 2\,e^2\,(1 - e^2)\cos^2\phi\cdot N}$$

$$= -\frac{1 - e^2 + e^2\cos^2\alpha\cos^2\phi}{N(1 - e^2)}$$

$$= -\frac{(1 - e^2)\,(\sin^2\alpha + \cos^2\alpha) + e^2\cos^2\alpha\,(1 - \sin^2\phi)}{N\,(1 - e^2)}$$

$$= -\frac{(1 - e^2)\sin^2\alpha + \cos^2\alpha - \cos^2\alpha\cdot e^2\sin^2\phi}{N\,(1 - e^2)} \times \frac{R_m}{R_m}$$

$$= -\frac{R_m\sin^2\alpha + \dfrac{R_m}{1 - e^2}\cos^2\alpha(1 - e^2\sin^2\phi)}{NR_m}$$

$$= -\frac{R_m\sin^2\alpha + N\cos^2\alpha}{NR_m}.$$

Substituting these differential coefficients in the usual formula for radius of curvature, we have*

$$R_\alpha = \frac{NR_m}{N\cos^2\alpha + R_m\sin^2\alpha}. \qquad [47]$$

If

$$\alpha = 0^\circ,$$

$$R_\alpha = \frac{NR_m}{N} = R_m,$$

the radius of curvature of the meridian; and if

$$\alpha = 90^\circ,$$

then

$$R_\alpha = \frac{NR_m}{R_m} = N,$$

the radius of curvature of the prime vertical.

Values of log R_α for different latitudes and azimuths will be found in Table XI. Observe that the value of R_α for 0° azimuth in this table equals that of R_m in Table X, and that for 90° azimuth equals N.

98. The Mean Value of R_α.

The mean value of R_α at any point on the spheroid for all azimuths from 0° to 360° may be found as follows: The mean value of any function of x between the limits a and b is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Substituting in this general equation we have

$$\text{mean } R_\alpha = \frac{1}{2\pi - 0}\int_0^{2} R_\alpha\,d\alpha$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{NR_m}{N\cos^2\alpha + R_m\sin^2\alpha}\cdot d\alpha$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{n}{N\cos^2\alpha + R_m\sin^2\alpha}\cdot d\alpha.$$

* Students who are familiar with Euler's Theorem, $\frac{1}{R_\alpha} = \frac{\sin^2\alpha}{N} + \frac{\cos^2\alpha}{R_m}$, will see that Equa. [47] follows at once from that theorem.

To integrate this expression it is necessary to change the variable. If we let

$$t = \tan\alpha\sqrt{\frac{R_m}{N}}, \quad \text{then} \quad dt = \sqrt{\frac{R_m}{N}} \cdot \frac{1}{\cos^2\alpha}.$$

By dividing both numerator and denominator by $N\cos^2\alpha$ and factoring NR_m the integral may be put in the form

$$\text{mean } R_\alpha = \frac{2}{\pi}\sqrt{R_mN}\int_0^{\frac{\pi}{2}} \frac{\sqrt{\frac{R_m}{N}} \cdot \frac{1}{\cos^2\alpha} \cdot d\alpha}{1 + \frac{R_m}{N}\frac{\sin^2\alpha}{\cos^2\alpha}}$$

which becomes

$$\text{mean } R_\alpha = \frac{2}{\pi}\sqrt{R_mN}\int_0^\infty \frac{dt}{1+t^2}$$

$$= \frac{2}{\pi}\sqrt{R_mN}\,[\tan^{-1} t]_0^\infty$$

$$= \sqrt{R_mN}. \qquad [48]$$

The mean radius of curvature is, therefore, the geometric mean of the radii of curvature of the two principal sections. The log of the mean radius is the arithmetical mean of the logs of the two radii.

99. Geometric Proof of Equa. [47].

Geometric proofs of the last two formulæ will be found instructive. To find $R\alpha$ geometrically, imagine a tangent plane at the point M and also a parallel plane at an infinitesimal distance below M. This second plane will cut from the surface a small ellipse. It has already been shown that the radius of curvature in the prime vertical plane is N. In Fig. 87 W, M, and E are three points on the circle of curvature whose radius is N and whose center is the point H on the axis. By similar triangles,

$$MC : CW = CW : CK.$$

Since MC is infinitesimal,

$$MC = \frac{a^2}{2N};$$

similarly, for three points in the plane of the meridian

$$MC = \frac{b^2}{2R_m}$$

and, in general, for any azimuth α,

$$MC = \frac{s^2}{2R_\alpha}.$$

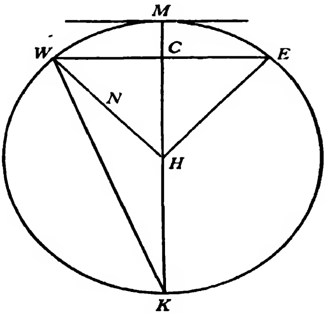

FIG. 87. Circle of Curvature.

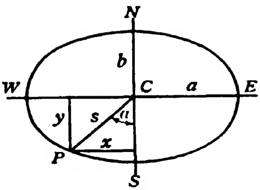

FIG. 88. Section of Ellipsoid.

The coördinates of the point P (Fig. 88) are

$$x = s \cdot \sin\alpha \quad \text{and} \quad y = s \cdot \cos\alpha.$$

Substituting these in the general equation of the ellipse,

$$\frac{s^2 \sin^2\alpha}{a^2} + \frac{s^2 \cos^2\alpha}{b^2} = 1.$$

But, from the preceding equations,

$$\frac{s^2}{a^2} = \frac{R_\alpha}{N} \quad \text{and} \quad \frac{s^2}{b^2} = \frac{R_\alpha}{R_m},$$

hence $$\frac{R_\alpha}{N} \cdot \sin^2\alpha + \frac{R_\alpha}{R_m} \cdot \cos^2\alpha = 1,$$

or $$R_\alpha = \frac{NR_m}{N\cos^2\alpha + R_m \sin^2\alpha}. \qquad [47]$$

99a. Geometric Proof of Equa. [48].

To show geometrically that the mean value of R_α $\sqrt{R_m N}$, observe that, as before,

$$\text{mean } R_\alpha = \frac{1}{2\pi}\int_0^{2\pi} R_\alpha \cdot d\alpha$$

and, from the preceding article,

$$R_\alpha = \frac{R_m s^2}{}$$

Therefore $$\text{mean } R_\alpha = \frac{1}{2\pi}\int_0^{2\pi} \frac{R_m s}{b^2} \quad d\alpha.$$

But $$\frac{1}{2}\int_0^{2\pi} s^2\, d\alpha = \text{area of ellipse} = \pi ab.$$

Therefore $$\text{mean } R_\alpha = \frac{1}{\pi} \times \pi ab \times \frac{R_m}{b^2} = \frac{aR_m}{b}$$

But $$\frac{a}{b} = \sqrt{\frac{N}{R_m}}.$$

Therefore $$\text{mean } R_\alpha = \sqrt{NR_m}. \qquad [48]$$

100. Length of an Arc of the Meridian.

Any small arc of the meridian ellipse may be regarded as an arc of a circle whose radius is R_m, the error being very small for short arcs. The length, therefore, is

$$s = R_m\, d\phi,$$

or, if $d\phi$ is in seconds of arc,

$$s = R_m\, d\phi'' \cdot \text{arc } 1''. \qquad [49]$$

If the arc is so long that the value of R_m varies appreciably, it is necessary to find s by integrating the expression

$$ds = \frac{a(1 - e^2)}{(1 - e^2 \sin^2\phi)^{\frac{3}{2}}} \cdot d\phi$$

between the limits ϕ_1 and ϕ_2.

If we expand the denominator by the binomial theorem, we have

$$ds = a(1 - e^2)(1 + \tfrac{3}{2} e^2 \sin^2\phi + \tfrac{15}{8} e^4 \sin^4\phi + \tfrac{35}{16} e^6 \sin^6\phi \cdots)\, d\phi.$$

Integrating,

$$s = a(1 - e^2)\int_{\phi_1}^{\phi_2}(1 + \tfrac{3}{2}e^2\sin^2\phi + \tfrac{15}{8}e^4\sin^4\phi + \tfrac{35}{16}e^6\sin^6\phi\cdot\cdot\cdot)\,d\phi.$$

In order to integrate the terms of the series in parenthesis we simplify the expression by means of the following relations:

$$\sin^2\phi = \tfrac{1}{2} - \tfrac{1}{2}\cos 2\phi,$$
$$\sin^4\phi = \tfrac{3}{8} - \tfrac{1}{2}\cos 2\phi + \tfrac{1}{8}\cos 4\phi,$$
$$\sin^6\phi = \tfrac{5}{16} - \tfrac{15}{32}\cos 2\phi + \tfrac{3}{16}\cos 4\phi - \tfrac{1}{32}\cos 6\phi.$$

Integrating and substituting the limits ϕ_1 and ϕ_2, and putting for abbreviation,

$$A = 1 + \tfrac{3}{4}e^2 + \tfrac{45}{64}e^4 + \tfrac{175}{256}e^6 = 1.005\,1093 \quad [0.0022133]$$
$$B = \tfrac{3}{4}e^2 + \tfrac{15}{16}e^4 + \tfrac{525}{512}e^6 = 0.005\,1202 \quad [7.709287]$$
$$C = \tfrac{15}{64}e^4 + \tfrac{105}{256}e^6 = 0.000\,0108 \quad [5.03342]$$
$$D = \tfrac{35}{512}e^6 = 0.000\,0000 \quad [2.326]$$

we obtain

$$s = a(1 - e^2)\big(A(\phi_2 - \phi_1) - \tfrac{1}{2}B(\sin 2\phi_2 - \sin 2\phi_1) + \tfrac{1}{4}C(\sin 4\phi_2 - \sin 4\phi_1) - \tfrac{1}{6}D(\sin 6\phi_2 - \sin 6\phi_1)\big). \quad [50]$$

100a. Area of a Quadrilateral on the Spheroid.

Two parallels of latitude separated by the distance $R_m\,d\phi$ lie on the surface of a cone and their circumferences are each $2\pi N\cos\phi$. The area of the enclosed strip is

$$dA = 2\pi N R_m \cos\phi\,d\phi$$
$$= 2\pi b^2\cos\phi\,d\phi\,(1 - e^2\sin^2\phi)^{-2}.$$

Expanding the last factor

$$(1 - e^2\sin^2\phi)^{-2} = 1 + 2e^2\sin^2\phi + 3e^4\sin^4\phi + 4e^6\sin^6\phi + 5e^8\sin^8\phi + \cdot\cdot\cdot,$$

we have

$$dA = 2\,\pi b^2 \cos\phi\, d\phi\, (1 + 2\, e^2 \sin^2\phi + \cdots).$$

If we integrate each term of the series the integral will be of the general form

$$\int \cos\phi \sin^n\phi\, d\phi = \frac{1}{n+1}\sin^{n+1}\phi.$$

The integration of the series gives

$$A_{\phi_1}^{\phi_2} = 2\, b^2\pi\left(\sin\phi + \tfrac{2}{3}e^2\sin^3\phi + \tfrac{3}{5}e^4\sin^5\phi + \tfrac{4}{7}e^6\sin^7\phi + \cdots\right)_{\phi_1}^{\phi_2}$$
$$= 2\, b^2\pi\left((\sin\phi_2 - \sin\phi_1) + \tfrac{2}{3}e^2(\sin^3\phi_2 - \sin^3\phi_1) + \cdots\right).$$

From trigonometry we have

$$\sin^3\phi = \tfrac{3}{4}\sin\phi - \tfrac{1}{4}\sin 3\,\phi$$
$$\sin^5\phi = \tfrac{5}{8}\sin\phi - \tfrac{5}{16}\sin 3\,\phi + \tfrac{1}{16}\sin 5\,\phi$$
$$\sin^7\phi = \tfrac{35}{64}\sin\phi - \tfrac{21}{64}\sin 3\,\phi + \tfrac{7}{64}\sin 5\,\phi - \tfrac{1}{64}\sin 7\,\phi$$

$$\therefore\quad A_{\phi_1}^{\phi_2} = 4\, b^2\pi\left(A\sin\left(\frac{\Delta\phi}{2}\right)\cos\phi_m - B\sin 3\left(\frac{\Delta\phi}{2}\right)\cos 3\,\phi_m\right.$$
$$\left. + C\sin 5\left(\frac{\Delta\phi}{2}\right)\cos 5\,\phi_m - D\sin 7\left(\frac{\Delta\phi}{2}\right)\cos 7\,\phi_m + \cdots\right)$$

in which

$$A = 1 + \frac{e^2}{2} + \tfrac{3}{8}e^4 + \tfrac{5}{16}e^6 + \tfrac{35}{128}e^8 = 1.003\ 4016 \quad [0.001\ 4748]$$
$$B = \tfrac{1}{6}e^2 + \tfrac{3}{16}e^4 + \tfrac{3}{16}e^6 + \tfrac{35}{192}e^8 = 0.001\ 1368 \quad [7.05568]$$
$$C = \tfrac{3}{80}e^4 + \tfrac{1}{16}e^6 + \tfrac{5}{64}e^8 = 0.000\ 0017 \quad [4.2304]$$
$$D = \tfrac{1}{112}e^6 + \tfrac{5}{256}e^8 = 0.000\ 0000$$

For a quadrilateral 1° on each side,

$$\Delta\phi = \phi_2 - \phi_1 = 1^\circ, \quad \text{and} \quad A_1 = A_{\phi_1}^{\phi_2} \div 360.$$

$$\therefore\quad A_1 = \frac{b^2}{90}\pi\,(A\sin 0^\circ\, 30'\cos\phi_m - B\sin 1^\circ\, 30'\cos 3\,\phi_m$$
$$+ C\sin 2^\circ\, 30'\cos 5\,\phi_m - D\sin 3^\circ\, 30'\cos 7\,\phi_m \cdots).$$

Values of these areas for 10′, 15′ and 30′ on a side will be found in "Geographic Tables and Formulas, S. S. Gannett, U. S. Geological Survey, 1904.

101. Miscellaneous Formulas.

The following formulas, relating to the ellipse, are given here for convenience of reference.

The geocentric latitude may be found from the expression

$$\tan \psi = \frac{y}{x} = (1 - e^2) \tan \phi = \frac{b^2}{a^2} \tan \phi. \qquad [51]$$

The maximum difference between ϕ and ψ is about 0° 11′ 40″, at latitude 45°. At the equator and at the poles the difference is zero.

The reduced latitude, θ (see Art. 94, p. 166), may be found from the geodetic latitude by means of the relation

$$a \tan \theta = b \tan \phi \qquad [52]$$

which is readily derived as follows:

Let MN (Fig. 84) $= y$, and $mN = Y$. For the ellipse,

$$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1.$$

For the circle

$$\frac{Y^2}{a^2} + \frac{x^2}{a^2} = 1.$$

Subtracting,

$$\frac{Y^2}{a^2} = \frac{y^2}{b^2} \quad \text{whence} \quad \frac{Y}{y} = \frac{a}{b} \quad \text{or} \quad \frac{mN}{MN} = \frac{a}{b}.$$

For the diagram,

$$\tan \theta = \frac{mN}{ON} \quad \text{and} \quad \tan \psi = \frac{MN}{ON}$$

$$\therefore \quad \frac{mN}{MN} = \frac{\tan \theta}{\tan \psi} = \frac{a}{b}.$$

But

$$\frac{b}{a} \tan \theta = \tan \psi = \frac{b^2}{a^2} \tan \phi \qquad \text{by [51]}$$

from which

$$a \tan \theta = b \tan \phi. \qquad [52]$$

The compression of the spheroid, that is, the flattening at the poles, is expressed by

$$f = a - b \qquad [53]$$

The length of a quadrant of the meridian is given by

$$q = \frac{a\pi}{2}\left(1 - \frac{e^2}{4} - \frac{3}{64}e^4 - \cdot \cdot \cdot \right).^* \qquad [54]$$

102. Effect of Height of Station on Azimuth of Line.

Since the normals drawn from two points on the surface do not

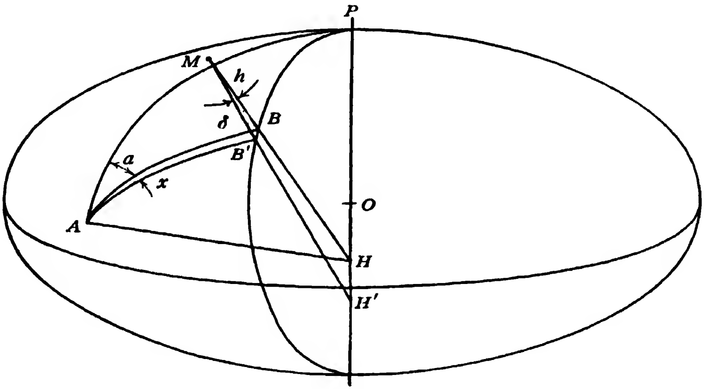

Fig. 89.

in general lie in the same plane, there will be an error in the observed horizontal direction of a station, depending upon its height above the surface of the spheroid. This error may be

* From the equation for the length of a meridian arc, we have for the quadrant

$$q = a(1 - e^2)\int_0^{\frac{\pi}{2}} (1 + \tfrac{3}{4}e^2(1 - \cos 2\phi) + \tfrac{15}{64}e^4(3 - 4\cos 2\phi + \cos 4\phi))\,d\phi$$

$$= a(1 - e^2)\left[\phi(1 + \tfrac{3}{4}e^2 + \tfrac{45}{64}e^4) - \tfrac{3}{8}e^2 \sin 2\phi - \tfrac{15}{32}e^4 \sin 2\phi + \tfrac{15}{256}e^4 \sin 4\phi\right]_0^{\frac{\pi}{2}}$$

$$= a(1 - e^2)\left[\frac{\pi}{2}(1 + \tfrac{3}{4}e^2 + \tfrac{45}{64}e^4 + \cdot \cdot \cdot)\right]$$

$$= \frac{a\pi}{2}(1 - \tfrac{1}{4}e^2 - \tfrac{3}{64}e^4 - \cdot \cdot \cdot).$$

likened to the error of sighting on an inclined range-pole; the higher up the sight is taken, the greater the error in the horizontal angle. In Fig. 89 the observer is at A and sighting at point M, which is at an elevation h above sea-level. The vertical plane of the instrument projects M down to sea-level at B on the line MH, H being the end of the normal at A. The point which is vertically below M is B', as determined by the normal MH'. Denote by δ the angle HMH' or, what is nearly the same, HBH'. The

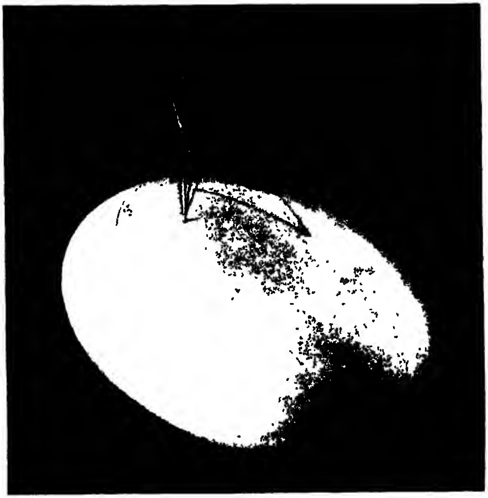

FIG. 90. A vertical in latitude 0° and a vertical in latitude 60°; $d\lambda = 80°$; $e = 0.81$; (looking SW).

angle (x) subtended by BB' at point A (the observer's position) is the correction desired. The latitude of A is ϕ, and that of M is ϕ'. In the triangle MHH'

$$\frac{\sin \delta}{\sin HH'M} = \frac{HH'}{HB + BM} = \frac{HH'}{HB} \text{ (approx.)},$$

or

$$\delta = \frac{HH'}{HB} \cdot \cos \phi',$$

where ϕ' is the latitude of B'.

Now $$HH' = OH' - OH$$
$$= (N' - n') \sin\phi' - (N - n)\sin\phi$$
$$= N'e^2 \sin\phi' - Ne^2 \sin\phi. \qquad \text{(See Equa. [45].)}$$

Therefore

$$\delta = \frac{\cos\phi'}{N}(N'e^2 \sin\phi' - Ne^2 \sin\phi)$$
$$= \frac{ae^2\cos\phi'}{N}\left(\frac{\sin\phi'}{(1 - e^2\sin^2\phi')^{\frac{1}{2}}} - \frac{\sin\phi}{(1 - e^2\sin^2\phi)^{\frac{1}{2}}}\right)$$

by Equa. [44]. Clearing of fractions and expanding the radicals in the numerators,

$$\delta = \frac{ae^2\cos\phi'}{N}\left(\frac{\sin\phi'\left(1 - \frac{e^2}{2}\sin^2\phi\cdots\right) - \sin\phi\left(1 - \frac{e^2}{2}\sin^2\phi'\cdots\right)}{1 - e^2\sin^2\phi_m}\right)$$

$$\frac{ae^2\cos\phi'}{N}\left(\frac{(\sin\phi' - \sin\phi)\left(1 + \frac{e^2}{2}\sin\phi\sin\phi'\right)}{1 - e^2\sin^2\phi_m}\right)$$

$$\frac{ae^2\cos\phi'}{N}\left(\frac{2\cos\phi_m\sin\frac{\Delta\phi}{2}\left(1 + \frac{e^2}{2}\sin\phi\sin\phi'\right)}{1 - e^2\sin^2\phi_m}\right)$$

$$\frac{ae^2\cos\phi'}{N}\cdot\cos\phi_m\cdot\frac{s\cos\alpha}{R_m}\cdot\left(\frac{1 + \frac{e^2}{2}\sin\phi\sin\phi'}{1 - e^2\sin^2\phi_m}\right)$$

$$\frac{ae^2\cos\phi'}{N}\cdot\cos\phi_m\cdot s\cos\alpha\cdot\frac{(1 - e^2\sin^2\phi)^{\frac{3}{2}}}{a(1 - e^2)}\left(\frac{1 + \frac{e^2}{2}\sin\phi\sin\phi'}{1 - e^2\sin^2\phi_m}\right)$$

$$\frac{e^2\cos^2\phi'\cdot s\cdot\cos\alpha}{N(1 - e^2)}\quad(1 - e^2\sin^2\phi)^{\frac{1}{2}}\left(1 + \frac{e^2}{2}\sin\phi\sin\phi'\right)$$

$$= \frac{e^2\cos^2\phi'\cdot s\cdot\cos\alpha}{N(1 - e^2)} + e^4 \text{ term (negl.).} \qquad (a)$$

The linear distance $BB' = h\delta$, and the correction to the

azimuth (x) at point A is given by

$$x'' = \frac{h\delta \sin \alpha}{s \text{ arc } 1''}$$

$$= \frac{hc^2 \cos^2 \phi' \cos \alpha \sin \alpha}{N (1 - e^2) \text{ arc } 1''} \qquad [55]$$

$$= k \frac{h}{N} \cos^2 \phi' \sin 2\alpha \qquad [56]$$

in which $$k = \frac{e^2}{2 (1 - e^2) \text{ arc } 1}$$

The logarithm of k for latitude $45° = 2.84685$.

When the signal is northeast or southwest of the observer the azimuth must be increased to obtain the correct azimuth at sea-level; if the signal is northwest or southeast the observed azimuth must be decreased.

For $\phi = 45°$, $\alpha = 45°$, and $h = 1000$ meters, the value of $x'' = 0''.055$. This is much smaller than the probable error of an observed direction (see p. 97), and is therefore negligible except for great heights. This correction has been applied to angles measured in the main triangulation of the California and Texas arc and the California and Washington arc. It is too small to affect the triangulation of the eastern half of the United States.

103. Refraction.

Inasmuch as the refraction acts in the vertical plane at any point, and the vertical plane changes its direction as the ray proceeds along the line, it is evident that there must be some horizontal displacement of the object sighted, due to the refraction. Investigations show that this error is quite inappreciable for all lines that can actually be observed.

104. Curves on the Spheroid. The Plane Curves.

When a theodolite is set up at any point A and leveled, its vertical axis is made to coincide with the direction of the force of gravity at A, which, except for local deflections, coincides with the direction of the normal at A. If another theodolite is set up

at B, in a different latitude and a different longitude, it is evident that these vertical axes are not in the same plane, since their normals (plumb lines) never intersect. The greater the latitude, the lower the point where the normal intersects the polar axis. It is clear that the line marked out on the surface of the spheroid by the line (or, rather, plane) of sight of the first theodolite is not the same as the line marked out by the vertical plane of the other theodolite. If A is southwest of B, then the curve cut by the plane of the theodolite at A is south of that cut by the plane of sight of the theodolite at B. This may be seen from the fact that both planes contain the chord AB; and since the normal at A is higher at the polar axis, the curve itself must be lower (farther south).

105. The Geodetic Line.

Another curve which holds an important place in the theory of geodesy is known as the *geodetic line*. This is the shortest line that can be drawn on the surface of the spheroid between two given points. It is not a plane curve, but has a *double curvature*. A characteristic property of the curve is that the osculating plane* at any point on the curve contains the normal to the surface at that point. In most cases the geodetic line is found to lie between the two plane curves and has a reversed curvature. Figure 91 is a photograph of a model, the semi-axes of which are $a = 6$ inches and $b = 3.5$ inches. The two plane curves are shown and between them, with the curvature slightly exaggerated, is the geodetic line.

In order to obtain a clear conception of the nature of the geodetic line, let us imagine that a transit instrument is set at point A (Fig. 92), leveled, and then sighted at point B. Then it is moved to point B, set up, and leveled again, and a back sight is taken on A; point C is then fixed by reversing the telescope. When the sight is taken to A, the sight line traces out the plane

* The osculating plane may be considered to pass through three consecutive points of the curve. In reality it is the limiting position approached by the plane as the distance between the three points decreases indefinitely.

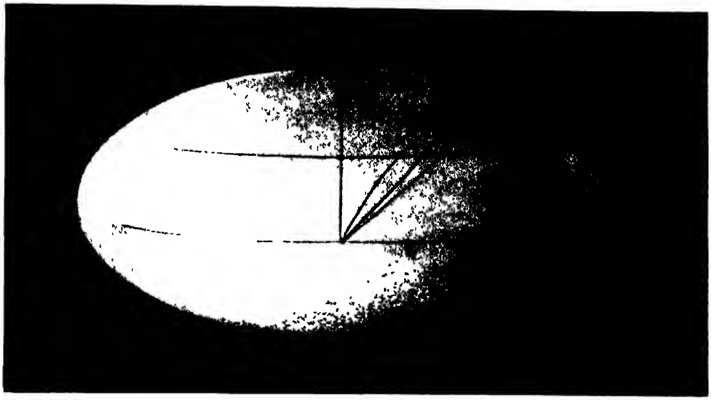

FIG. 91. Plane Curves and Geodetic Line.

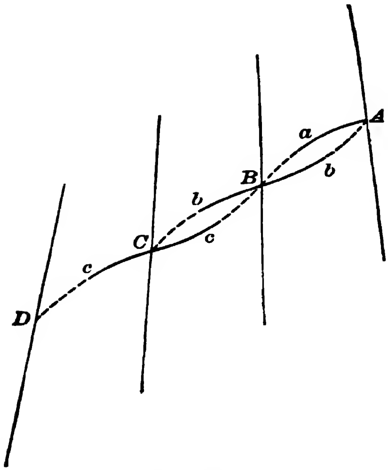

FIG. 92.

curve BbA; and when point C is sighted, it traces out BbC. The instrument is then taken to C and the process repeated. It should be observed that the (vertical) sight plane of the instrument coincides with the normal to the surface at each station. If the points A, B, C, D are imagined to approach nearer and nearer, so that AB, BC, etc., become infinitesimal elements of the curve, the plane which contains three consecutive points of the curve also contains the normal to the surface. If we imagine the instrument to move along this line, it is seen that the vertical plane of the instrument twists so that it always contains the normal.

One of the characteristic properties of the geodetic line is shown by the equation

$$R_p \sin \alpha = k, \text{ a constant} \qquad [57]$$

R_p being the radius of the parallel and α the azimuth of the geodetic line at any point. This equation may be derived analytically by the methods of the calculus of variations (see Clarke, *Geodesy*, p. 125) or by geometric construction (see Jordan, *Vermessungskunde*, Vol. III, p. 395). From this equation it will be seen that when α is a maximum (90°), $\sin \alpha = 1$ and $R_p = k$. The constant of the equation is therefore the radius of the parallel of latitude beyond which the geodetic line does not pass. When α is a minimum, R_p is a maximum, that is, $R_p = a$, the equatorial radius of the spheroid. This shows that in general a geodetic line cutting the equator at any angle α may go northward up to some (limiting) parallel of latitude $\phi°$ (corresponding to $R_p = k$), but will not pass north of this parallel. In the southern hemisphere it will reach a limit $(-\phi°)$ having the same numerical value. Such a geodetic line, when traced completely around the spheroid, will not in general return exactly on itself, but will pass the initial point on the equator in a slightly different longitude and then proceed to form another loop around the spheroid.

Except for a few particular cases the geodetic line lies between

the two plane curves and divides the angle between them in the ratio of about 2 to 1, as shown in Fig. 93.

If the terminal points P and Q are in nearly the same latitude, the geodetic line may cross the plane curve.

It is important to bear in mind that the lengths of these different curves on the spheroid differ by quantities that are quite inappreciable in practice. The differences in length are far shorter than the distances by which the curves are separated at their middle points (Art. 107), and even these latter are negligible in practice. Also the angle by which the azimuth of the geodetic

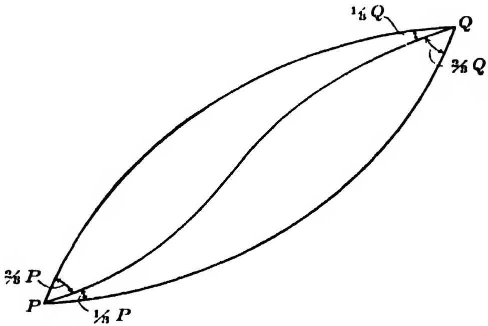

FIG 93. The Plane (elliptical) curves and the Geodetic Line.

differs from the azimuth of the plane section is much smaller than can be measured.

It should be noted that the geodetic line itself cannot be sighted over directly, because it is not a plane curve, and that the *geodetic triangle* can be obtained only by computation.

106. The Alignment Curve.

Another curve which may be drawn on the surface is defined in the following manner: if the theodolite be supposed to move from A to B, keeping always in line between the two points (that is, the azimuths of A and B 180° apart), and the instrument being always leveled, its path will be a curve which lies very close to

the geodetic line and generally between the two plane curves. This is called the *alignment curve*.

It is possible to define other curves* between these two points. Such curves are of theoretical value only, since the lengths of all such lines on the earth's surface differ from each other by quantities too small to measure. The two plane curves, however, are separated by a distance which is sometimes quite appreciable.

107. Distance between Plane Curves.

The maximum separation of the two plane curves may be computed approximately as follows: the angle (δ') between the

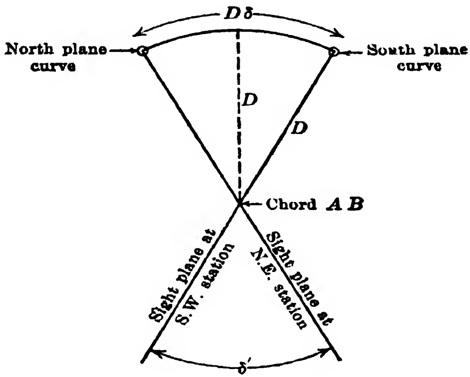

FIG. 94.

two planes is very nearly equal to the angle δ multiplied by sin α, since δ is the angle measured in the plane of the meridian, whereas the angle desired (δ', Fig. 94) is that perpendicular to the planes of sight. Therefore

$$\delta' = \frac{se^2 \cos^2 \phi \cos \alpha \sin \alpha}{N(1 - e^2)}$$

(see Equa. (a), p. 184).

* See Coast Survey Report for 1900, p. 369

The distance of the chord AB (Fig. 95) below the surface (D) at its middle point is given by

$$D : \frac{s}{2} = \frac{s}{2} : 2\,R_\alpha,$$

or, approximately,

$$D = \frac{s^2}{8\,N}.$$

The curves are separated at their middle points by the horizontal distance

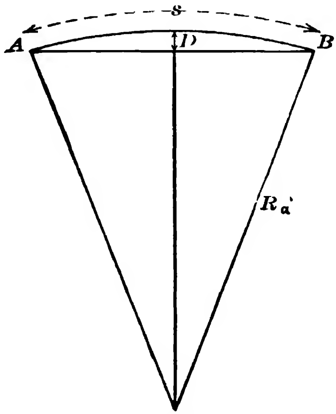

FIG. 95.

$$D\delta = \frac{s^2}{8\,N} \times \frac{se^2 \cos^2 \phi \cos \alpha \sin \alpha}{N\,(1 - e^2)}$$

$$= \frac{s^3\,e^2 \cos^2 \phi\, \cos \alpha \sin \alpha}{8\,N^2\,(1 - e^2)}. \qquad [58]$$

The difference in azimuth may be computed approximately by finding the angle between the two tangents to the curve drawn from one of the stations and prolonged half the distance (Fig. 96). The terminal points of these tangents will be at a distance

D above the surface and will be separated by a distance $2\,D\delta'$. The angle between these two lines is nearly

$$= \frac{2\,D\delta'}{\frac{1}{2}\,s \text{ arc } 1''}$$

$$\frac{2\,s^3}{8\,N^2} \cdot \frac{e^2 \cos^2\phi \cos\alpha \sin\alpha}{(1 - e^2)\,\frac{1}{2}\,s \text{ arc } 1''}$$

$$\frac{s^2}{2\,N^2} \cdot \frac{e^2 \cos^2\phi \cos\alpha \sin\alpha}{(1 - e^2) \text{ arc } 1''}. \qquad [59]$$

For the oblique boundary line between California and Nevada* $s = 650{,}000$ m., (400 mi.), $\phi_m = 37°\ 00'$, $\alpha = 134°\ 33'$; whence $D\delta' = 1.8$ meters and the difference in azimuth $= 2''.3$.

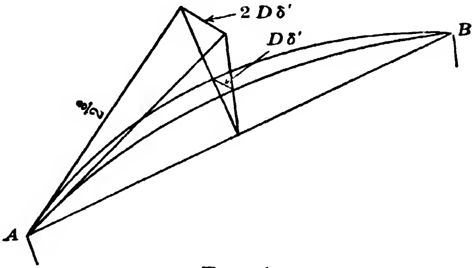

Fig 96.

For the western boundary of Massachusetts $s = 80{,}930$ m., (50 mi.), $\phi_m = 42°\ 24'$, $\alpha = 195°\ 12'$; this gives $D\delta' = 0.0015$ meter and $\Delta\alpha = 0''.016$.

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PROBLEMS

Problem 1. What is the distance in miles from the center of the ellipsoid to the plumb line (normal) at New York (Latitude 40° 42′ N.)?

* See Coast Survey Report for 1900, p. 368.

Problem 2. If a section of the ellipsoid is cut by a plane one millimeter below and parallel to a plane which is tangent to the surface in latitude 40°, what are the semi-axes of this ellipse?

Problem 3. Compute the length in meters of a quadrant of the Clarke Spheroid of 1866. (See pp 182 and 409).

Problem 4. Compute the ellipticity of the Clarke spheroid.

Problem 5. If a sphere whose radius is $\sqrt{R_m N}$ is tangent to the spheroid in $\phi = 45°$ how far is the spherical surface from the spheroidal surface at a distance of 160 kilometers from the point of tangency. (*a*) in the plane of the meridian. (*b*) in the east-west plane.

Problem 6. What is the area in sq. meters between the 40° and 41° parallels and two meridians 1° apart?

Problem 7. What is the error in the direction of a sight taken on Pike's Peak, whose altitude is 14,108 feet? (Art. 102.)

CHAPTER VI

CALCULATION OF TRIANGULATION

108. Preparation of the Data.

From the records of the field-work of the triangulation we obtain a value for each angle, supposed to be freed from the errors of the instrument, eccentricity of station, phase of signal, elevation of signal, etc. Before these angles are employed for solving the triangles, they should be examined to see if they satisfy any geometric conditions existing among them. If at any station two or more angles and their sum have been measured, then these angles must be so corrected that they exactly equal their sum. If the horizon has been *closed*, the measured angles must be adjusted so that their sum equals 360°. If the angles have been measured with different degrees of precision, as, for example, with different instruments or a different number of sets or of repetitions, the different angles should be given proper weights; and if the best possible values are desired, the angles at each station should be adjusted by the method of least squares.

After the *station adjustment*, as it is called, has been completed, the triangles must be examined to see if the sum of the three angles in each triangle fulfills the requirement that this sum shall equal 180° plus the spherical excess of the triangle. The verticals at the three triangulation stations are not parallel to each other, because the surface is curved. Consequently the sum of the angles will exceed 180° by an amount which, on a spherical surface, would be exactly proportional, and which, on a spheroidal surface, is nearly proportional to the area of the triangle.

As was shown in the preceding chapter (Art. 102), the error in the direction of an object, due to the fact that the earth is spheroidal

instead of spherical, is extremely small, even when the object is several thousand meters above sea-level. Hence it follows that if the vertices of a spheroidal triangle are projected vertically onto the surface of a tangent sphere,* the errors thus produced in the horizontal angles of the triangle will be much less than the errors in the measurement of the angles, because the points on the sphere and those on the spheroid are separated by comparatively short distances. This enables us to compute spheroidal triangles as spherical triangles and greatly simplifies the computation. The lengths of the triangle sides will be practically the same on the two surfaces.

In this connection it is well to bear in mind that if the topography of the earth's surface were represented on an 18-inch globe the total variation in elevation would scarcely be greater than the thickness of a coat of varnish. The elevation of the geoid above the spheroid would be very much smaller than this, and the distance between the spheroid and the tangent sphere at any station would usually be still smaller. This will give some idea of the minuteness of the errors under discussion.

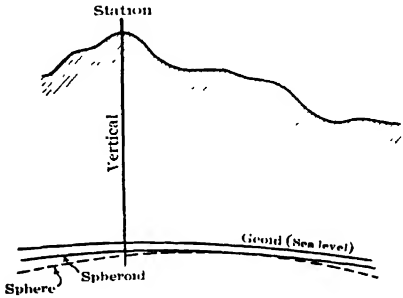

FIG. 97.

It should be remembered that, whereas the triangulation stations themselves are at various heights above sea-level, these are all supposed to have been projected down vertically onto the spheroid before beginning the computation of the triangle. The points of which we shall speak in discussing the solution of the triangles and the geographical positions of the stations are these points on the spheroidal surface and not the original station points.

* The sphere is supposed to be tangent at the center of gravity of the triangle to be computed.

In solving triangles by the methods given below, the following approximations have been made, and it is assumed that the resulting errors are negligible.

1. The reduction to sea-level reduces the observed direction to that corresponding to the geoid (or actual surface), not the spheroid, as is assumed.

2. The effect of local deflection of the plumb line is not usually allowed for. In some cases, however, it becomes appreciable.

3. The effect of atmospheric refraction on the direction (horizontal component of the refraction) is neglected.

4. The reduction of the observed direction (plane curve) to that of the geodetic, or shortest, line is omitted. There are in reality eight triangles formed by the plane curves, which are treated as if they were identical (see Art. 104).

109. Solution of a Spherical Triangle by Means of an Auxiliary Plane Triangle.

The direct solution of the triangles of a net as spherical triangles would be unnecessarily complicated. This may be avoided by employing a principle known as Legendre's Theorem, namely, that if we have a spherical triangle whose sides are short compared with the radius of the sphere, and also a plane triangle whose sides are *equal in length* to the corresponding sides of the spherical triangle, then the corresponding angles of the two triangles differ by approximately the same quantity, which is one-third of the spherical excess of the triangle.

110. Spherical Excess.

The spherical excess of a triangle is directly proportional to its area, as shown in spherical geometry. Hence, if A' is the area of any triangle, R is the radius of the sphere, S is the surface of the sphere, and e is the spherical excess of the triangle; then, since the spherical excess of the tri-rectangular triangle is $\frac{\pi}{2}$,

$$\frac{e}{\frac{\pi}{2}} = \frac{A'}{\frac{1}{8}S},$$

or
$$\frac{2e}{\pi} = \frac{2A'}{\pi R^2}.$$

Therefore
$$e = \frac{A'}{R^2}.$$

To express e in seconds of arc, divide by arc 1″, and we have

$$e'' = \frac{A'}{R^2 \text{ arc } 1''} = \frac{bc \sin A}{2 R^2 \text{ arc } 1''}, \qquad [60]$$

where b, c, and A are two sides and the included angle of the plane triangle, b and c being in linear units.

The sphere which is tangent to the spheroid at the center of gravity of the triangle, and which has the same average curvature, is a sphere of radius $= \sqrt{R_m N}$; whence

$$e'' = \frac{bc \sin A}{2 R_m N \text{ arc } 1''} = mbc \sin A. \qquad [61]$$

The log of the quantity $\frac{1}{2 R_m N \text{ arc } 1''} = m$ is given for different latitudes in Table XII. The latitude to be used in finding m is the mean of the latitudes of the three vertices of the triangle.

Questions. — Is this auxiliary plane triangle the same as the chord triangle formed by joining the points by straight lines? Are the two similar in shape?

The formulæ for e'' given in Equas. [60] and [61] are sufficiently accurate for all triangles except a few of the very largest, such as those occurring in the Davidson Quadrilaterals in California and Nevada. When the sides of the triangle are over 100 miles in length it may become necessary to use the following more accurate formula:*

$$e_1'' = \frac{\text{Area}}{R^2 \sin 1''}\left(1 + \frac{a^2 + b^2 + c^2}{24 R^2}\right) \qquad [61a]$$

or,
$$e_1'' = e'' + e'' \times \frac{a^2 + b^2 + c^2}{24 R^2}$$

For proof see Appendix.

111. Proof of Legendre's Theorem.

To prove Legendre's theorem, let A', B' and C' be the angles of the spherical triangle, and A, B, and C those of the plane triangle; the lengths of the sides of the plane triangle are a, b, and c, and those of the spherical triangle are $a'R$, $b'R$, and $c'R$, then, in the plane triangle,

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc} \tag{a}$$

or

$$\sin^2 A = 1 - \cos^2 A = \frac{4\,b^2c^2 - (b^2 + c^2 - a^2)^2}{4\,b^2c^2}.$$

$$\frac{2\,a^2b^2 + 2\,a^2c^2 + 2\,b^2c^2 - a^4 - b^4 - c^4}{4\,b^2c^2} \tag{b}$$

In the spherical triangle,

$$\cos A' = \frac{\cos a' - \cos b' \cos c'}{\sin b' \sin c'}.$$

Expanding each sine and cosine (omitting terms of higher order than the fourth),

$$\cos A' = \frac{1 - \frac{a'^2}{2} + \frac{a'^4}{24} - \left(1 - \frac{b'^2}{2} + \frac{b'^4}{24}\right)\left(1 - \frac{c'^2}{2} + \frac{c'^4}{24}\right)}{\left(b' - \frac{b'^3}{6}\right)\left(c' - \frac{c'^3}{6}\right)}$$

$$= \frac{\frac{1}{2}(-a'^2 + b'^2 + c'^2) - \frac{1}{24}(b'^4 + c'^4 - a'^4) - \frac{1}{4}\,b'^2c'^2}{b'c'\,[1 - \frac{1}{6}(b'^2 + c'^2)]}$$

$$= [\tfrac{1}{2}(-a'^2+b'^2+c'^2) - \tfrac{1}{24}(b'^4+c'^4-a'^4) - \tfrac{1}{4}\,b'^2c'^2]\,\frac{1 + \frac{1}{6}(b'^2+c'^2)}{b'c'}$$

$$\frac{b'^2 + c'^2 - a'^2}{2\,b'c'} \quad \frac{b'^4 + c'^4 - a'^4 + 6\,b'^2c'^2}{24\,b'c'}$$

$$+ \frac{-a'^2b'^2 + b'^4 + 2\,b'^2c'^2 - a'^2c'^2 + c'^4}{12\,b'c'}$$

whence $\cos A' =$

$$\frac{b'^2 + c'^2 - a'^2}{2\,b'c'} \quad 1 \quad \frac{2\,a'^2b'^2 + 2\,a'^2c'^2 + 2\,b'^2c'^2 - a'^4 - b'^4 - c'^4}{4\,b'c'} \tag{c}$$

From (a), (b), and (c)

$$\cos A' = \cos A - \tfrac{1}{6}\, b'c' \sin^2 A.$$

Let x be the difference between A and A'. Then

$\cos x = 1$ and $\sin x = x'' \text{ arc } 1''$ (nearly), since x is small,

and
$$\begin{aligned} \cos A' &= \cos (A + x) \\ &= \cos A - \sin A x'' \text{ arc } 1'' \\ &= \cos A - \tfrac{1}{6}\, b'c' \sin^2 A; \end{aligned}$$

that is,
$$x'' \text{ arc } 1'' \sin A = \tfrac{1}{6}\, b'c' \sin^2 A.$$

Therefore
$$x'' = \frac{b'c' \sin A}{6 \text{ arc } 1'}$$

or, since
$$b' = \frac{b}{R} \quad\text{and}\quad c' = \frac{c}{R},$$

$$x'' = \frac{bc \sin A}{6\, R^2 \text{ arc}} \qquad [62]$$

It will be noticed that this is one-third of the spherical excess as found in Equa. [60]. The same result would also be found for angles B and C.

112. Error of Legendre's Theorem.

The error in Legendre's theorem* as applied to the sphere may be studied by carrying out the above series so as to include terms of higher powers than the fourth. Jordan (*Vermessungskunde*) gives a numerical example showing the amount of this error in a triangle of which the side AC is about 65 miles in length; the angles are shown below:

$A' =$	40°	39′	30″.380
$B' =$	86	13	58 .840
$C' =$	53	06	45 .630
	180°	00′	14″.850

Denoting the spherical angles by A', B', C', and the corresponding plane angles by A, B, C, the differences are as follows, the first column containing the values derived from Legendre's

* See Coast Survey Special Publication No. 4, p. 51.

theorem in its ordinary form, the second containing the smaller terms which are usually neglected.

	Approx.	Exact.
$A' - A$	4″.950018	4.950036
$B' - B$	4 .950018	4.949997
$C' - C$	4 .950018	4.950021

113. Calculation of Spheroidal Triangles as Spherical Triangles. It is customary to assume that the differences between the spherical and spheroidal triangles are negligible when the actual points are projected down onto a tangent sphere of radius $\sqrt{R_m N}$. Clarke, in his *Geodesy*, shows the error of this assumption in the case of a triangle having a side over 200 miles long, the result being as follows:

	Spheroidal		Spherical
A'	98° 44′ 37″.0965	A	98° 44′ 37″.1899
B'	58° 16′ 46″.5904	B	58° 16′ 46″.4737
C'	23° 00′ 12″.7303	C	23° 00′ 12″ 7034
e'	1′ 36″.4262	e	1′ 36″ 4270

The preceding example indicates that in triangles whose sides are lines that can be sighted over on the earth's surface the error involved in computing spheroidal triangles as spherical triangles is negligible in practice.

Jordan, Vol. III, p. 579, gives for the triangle mentioned in Art. 112,

$$\begin{aligned} A &= 4''.950\ 184 \\ B &= 4\ .949\ 969 \\ C &= 4\ \ 949\ 901 \\ \hline e &= 14''.850\ 054 \end{aligned}$$

These are the seconds of the angles corrected for spheroidal excess; this shows that for ordinary triangles the differences between spherical and spheroidal angles are beyond the thousandths place of the seconds.

114. Calculation of the Plane Triangle.

After the spherical excess has been computed, the angles of an auxiliary plane triangle may be found by applying Legendre's

theorem, that is, by deducting one-third of the spherical excess from each spherical angle. The difference between the sum of these plane angles and 180° is the error of measurement and may be distributed equally among the three angles unless a least-square adjustment is to be made. In any case this method of distributing the error may be used for a preliminary determination of the distances. The lengths of the triangle sides are now found by plane trigonometry. Since all three angles of a triangle will usually be known, the only formula that will be used, except in rare cases, is the sine formula,

$$\frac{a}{b} = \frac{\sin A}{\sin B}.$$

or, $$\log a = \log b + \log \operatorname{cosec} B + \log \sin A.$$

A convenient arrangement of this computation, used by the Coast and Geodetic Survey, is shown in the following table. The spherical excess of the triangle in this case is 0″.86, which gives 1″.24 as the error of closure of the triangle.

Stations.	Observed angles.	Corrections.	Spherical angles.	Spherical excess.	Plane angles and distances.	Logarithms.
Blue Hill to Prospect					22723.08 *m*	4 356 4673
	° ′ ″	″	″	″	° ′ ″	
Observatory	61 47 18 80	−0 41	18 39	−0 29	61 47 18 10	0 054 9218
Blue Hill	35 45 15 40	−0 41	14 99	−0 29	35 45 14 70	9.766 6415
Prospect	82 27 27 90	−0 42	27 48	−0 28	82 27 27 20	9 996 2261
	180 00 02 10	−1 24	00 86	−0 86	180 00 00 00	
Observatory to Prospect					15067 13 *m*	4.178 0306
Observatory to Blue Hill					25563 20 *m*	4 407 6152

115. Second Method of Solution by Means of an Auxiliary Plane Triangle.*

Another method of solution which has been used to some extent in Europe is as follows:

* See Jordan, *Vermessungskunde*, Vol. III, § 39.

Let ABC (Fig. 98) be the spherical triangle and $A'B'X$ an auxiliary plane triangle having two of its angles, α and β, equal to the corresponding angles in the spherical triangle. Evidently the third angles will not be equal.

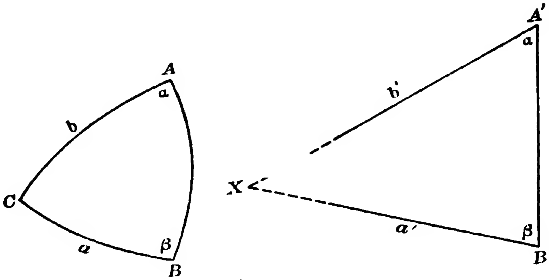

Fig. 98.

Let a' and b' in the plane triangle be the sides corresponding to a and b.

In the spherical triangle we have

$$\frac{\sin\alpha}{\sin\beta} = \frac{\sin\frac{a}{R}}{\sin\frac{b}{R}}$$

and in the plane triangle

$$\frac{\sin\alpha}{\sin\beta} = \frac{a'}{b'}$$

for all values that may be given to a' and b'; whence

$$\frac{\sin\frac{a}{R}}{\sin\frac{b}{R}} = \frac{a'}{b'} = \frac{\frac{a'}{R}}{\frac{b'}{R}}$$

This equation is satisfied if we place

$$\frac{a'}{R} = \sin\frac{a}{R}$$

and

$$\frac{b'}{R} = \sin\frac{b}{R}.$$

The general expression for any triangle side may be written

$$\frac{s'}{R} = \sin\frac{s}{R},$$

s' being the side of an auxiliary plane triangle corresponding to the side s of the spherical triangle.

Taking logs of both members,

$$\log\frac{s'}{R} = \log\sin\frac{s}{R} = \log\left(\frac{s}{R} - \frac{s^3}{6\,R^3} + \frac{s^5}{120\,R^5}\cdots\right)$$

$$\log\frac{s}{R} + \log\left(1 - \frac{s^2}{6\,R^2} \qquad \cdot\right).$$

Now, since

$$\log(1 + x) = M\left(x - \frac{x^2}{2} + \frac{x^3}{3}\cdots\right)$$

(where $M = \log_{10} e = 0.4342945$, the modulus of the common logarithms), we may write

$$\log\frac{s'}{R} = \log\sin\frac{s}{R} = \log\frac{s}{R}$$

$$+ M\left(-\frac{s^2}{6\,R^2} + \frac{s^4}{120\,R^4}\cdots\right) - \frac{M}{2}\left(-\frac{s^2}{6\,R^2}\right)^2\cdots$$

$$= \log\frac{s}{R} - \frac{Ms^2}{6\,R^2}$$

Therefore $$\log\frac{s}{R} - \log\frac{s'}{R} = \frac{Ms^2}{6\,R^2},$$

or $$\log s - \log s' = \frac{Ms^2}{6\,R^2}, \qquad [63]$$

which is the correction to the log of the triangle side.

In calculating this correction, R^2 should be replaced by $R_m N$. Values of these corrections will be found in Table XIII for the argument log s.

* The next term $= \dfrac{M}{180}\cdot\dfrac{s^4}{R^4} = 0.000\,000\,0001$ for a distance of 100 kilometers

Example.

Stations	Spherical angles.	Distances.	Logarithms.
Blue Hill to Prospect	° ′ ″	22,723.08	4.356 4673
Correction			9
s' .			4.356 4664
Observatory. . .	61 47 18.39		0.054 9215
Blue Hill. .	35 45 14.99		9.766 6423
Prospect . . .	82 27 27.48		9.996 2262
s' . . .			4.178 0302
Correction . . .			4
Observatory to Prospect.		15,067.13	4.178 0306
s' .			4.407 6141
Correction			11
Observatory to Blue Hill.		25,563 20	4.407 6152

Notice that after the base of the first triangle has once been reduced by subtracting the correction, the computation of the whole chain of triangles may be carried out, using the spherical angles only. It is not necessary to add the corrections to the logarithms of the computed sides until their true values are to be found.

PROBLEMS

Problem 1. Compute the area in square miles of a triangle on the earth's surface having a spherical excess of 1″, assuming that the earth is a sphere of radius 3960 miles.

Problem 2. Compute the sides of the following triangles:

Station	Correction to angles from figure adjustment.	Error of closure of triangle.	Corrected spherical angles	Spherical excess.
(*a*) Mt. Ellen	−0″.70	+0″.22	49° 36′ 36″.88	34″.33
Tushar	+0 .98		55 56 26 .70	
Wasatch	−0 .06		74 27 30 .75	

Wasatch to Mt. Ellen; azimuth, 333° 01′ 08″.65; back-azimuth, 153° 25′ 05″.00; dist., 123,556.70 meters; logarithm, 5.0918663. Latitude of Wasatch, 39° 06′-54″.362; longitude, 111° 27′ 11″.915. (Tushar is SW of Wasatch.)

Station	Correction	Error of closure	Corrected spherical angles	Spherical excess
(*b*) Uncompahgre	+0″.17	+0″.65	31° 54′ 61″.57	46″.15
Mt. Waas	−0 .10		98 16 41 .16	
Tavaputs	+0 .58		49 48 63 .42	

Mt. Waas to Uncompahgre; azimuth, 288° 01′ 25″.71; back-azimuth, 109° 07′-06″.11; dist. 162,928.01 meters, logarithm, 5.211 9958. Latitude Mt. Waas, 38° 32′ 21″.444; longitude, 109° 13′ 38″.302. (Mt. Waas is SW of Tavaputs.)

Problem 3. Position of point *B* $\begin{cases} \text{lat. } 39° \ 13' \ 26''.686 \\ \text{long. } 98° \ 32' \ 30''.506 \end{cases}$

Position of point *C* $\begin{cases} \text{lat. } 38° \ 51' \ 50''.913 \\ \text{long. } 98° \ 29' \ 15''.508 \end{cases}$

Azimuth *B* to *C* 353° 17′ 21″.81; dist. 40232.35 meters; (log = 4.604 5754); back-azimuth 173° 19′ 24″.64.

The spherical angles are *A* 57° 53′ 14″.39 (*A* is east of *BC*.)
B 62° 23′ 31″.40
C 59° 43′ 17″.93

Compute the spherical excess and solve the triangle.

Problem 4. Position of pt. *L*; latitude 42° 26′ 13″.276, longitude 70° 55′ 52″.088. Distance *L* to *N*, 3012.0 meters (log = 3.478 8600). Azimuth *L* to *N*, 314° 34′ 00″; back-azimuth, 134° 35′ 03″. Position of pt. *N*, latitude 42° 25′ 04″ 764, longitude 70° 54′ 18″.232. Angle at *L*, 36° 15′ 07″; at *N*, 63° 44′ 50″; at *E*, 79° 59′ 57″. (*E* is east of *LN*.) Compute the spherical excess and solve the triangle.

Problem 5. The observed angles of a triangle and their corrections as found by adjustment are as follows:

	Angle	Corrections.
Sand Hill	40° 57′ 28″.13	−0″.35
Rutherford	54 22 59 .51	−0 .61
Miller	84 39 35 .03	−0 .44

The position of Rutherford is latitude = 37° 08′ 57″.928 N, longitude = 98° 06′-31″.618 W. The position of Miller is latitude = 37° 02′ 20″.963 N, longitude 97° 55′ 43″.908 W. The azimuth from Miller to Rutherford = 127° 28′ 17″.95; back-azimuth 307° 21′ 47″.30. Distance in meters, 20139.64; logarithm, 4 304 0518. Solve the triangle.

Problem 6. Show that the substitution of Equas. (*a*) and (*b*) p. 198 in Equa. (*c*) p. 198 is permissible under the assumptions made in Arts. 109 and 111.

Problem 7. The angles and sides of the Mt. Lola — Mt. Diablo — Mt. Helena triangle are as follows: —

Lola	28° 49′ 07″.900	107 728.96^m
Diablo	73 06 32 .540	213 873.23
Helena	78 05 16 .803	218 704.43

Compute the second term of the spherical excess. The mean latitude is 38° 40′ N.

Problem 8. The sides of the triangle Shasta — Helena — Lola are approximately 133 mi., 167 mi., and 190 mi. The value of e'' is 142″.696. Compute the second term of the spherical excess.

CHAPTER VII

CALCULATION OF GEODETIC POSITIONS

116. Calculation of Geodetic Positions.

In geodetic surveys covering large areas the positions of the triangulation points are expressed by means of their latitudes and longitudes. Over limited areas a system of rectangular spherical coördinates may be used to advantage, but for such areas as have to be surveyed in this country the latitude and longitude system is preferable.

Before the latitude and longitude of one triangulation station can be calculated from the coördinates of another station, it is necessary to know the dimensions of the spheroid which is taken to represent the earth's figure, and also to fix definitely the latitude and longitude of some specified station, as well as the azimuth of the direction to some other triangulation station. This selected position and direction determine the relative position of the whole survey with respect to the adopted spheroid, and constitute what is known as the *geodetic datum*. The surveys of different countries may be computed on different spheroids or may be located inconsistently on the same spheroid. The different portions of a survey of the same country will be located inconsistently on the same spheroid until they have been connected by triangulation.

The two spheroids which have been most extensively used for geodetic surveys are (1) that computed by Bessel in 1841, and (2) that by Clarke in 1866. The Bessel spheroid was computed from data obtained chiefly on the continent of Europe, and consequently conforms closely to the curvature of that portion of the earth. This spheroid is still in general use in Europe. Clarke's spheroid of 1866 was computed from arcs distributed over a much

larger portion of the earth's surface; it shows a greater amount of flattening at the poles than the Bessel spheroid, and consequently assigns a flatter curvature to the surface in the latitude of Europe and of the United States. The Bessel spheroid was employed by the Coast Survey in the earlier years. As the surveys gradually extended, the errors due to using this spheroid became more and more apparent, until finally, about 1880, it was decided to change to the Clarke spheroid. The latter conforms much more nearly to the curvature of the surface in the United States.

117. The North American Datum.*

In 1901 the United States Coast and Geodetic Survey adopted what was then called the United States Standard Datum, by assigning to the station *Meades Ranch* the following position on the Clarke spheroid:

Latitude,	39° 13′ 26″.686
Longitude,	98° 32′ 30″.506
Azimuth to *Waldo*,	75° 28′ 14″ 52

In 1913 this datum was adopted by the governments of Canada and Mexico, and it is now known as the North American Datum.

In deciding upon a geodetic datum it was necessary to consider two important points: first, the datum should be so chosen as to reduce to a minimum the labor of recomputing the geodetic positions; second, it must place the triangulation system in such a position that no serious error will occur in any part of the system. At the time this datum was selected a large number of triangulation points had already been located along the Atlantic Coast. By selecting a position for *Meades Ranch* consistent with the old datum upon which this triangulation was calculated, a large amount of recomputation was avoided. At the same time it was apparent that this also placed the triangulation very near to its theoretically best position.

* See Coast Survey Special Publication No. 24, p. 8, or Special Publication No. 19, p. 80.

118. Method of Computing Latitude and Longitude.

Assuming that the latitude and longitude of a station (A) are known, as well as the distance and azimuth to a second station (B), we will now develop the formulæ* necessary to compute the geodetic latitude and longitude of the second point. In doing this we shall have to solve the differential spherical triangle formed by joining the two points with the pole.

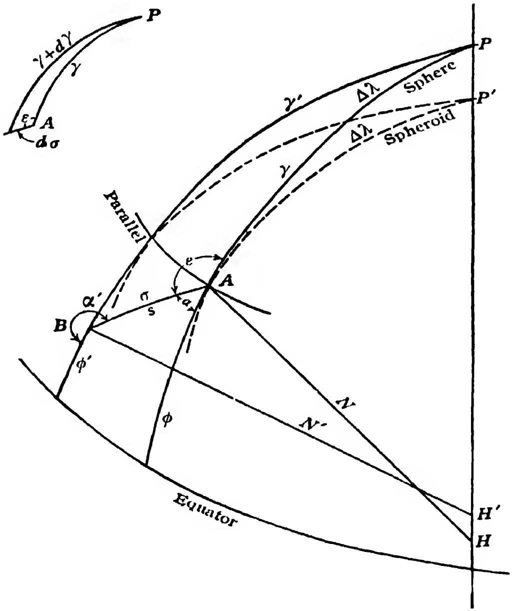

FIG. 99

119. Difference in Latitude.

In Fig. 99, P' is the pole of the spheroid. P is the pole of a sphere tangent to the spheroid along the parallel of latitude

* These formulæ were first given by Puissant; see his Traité de Géodesie, Vol. I; see also Coast and Geodetic Survey Report for 1894, and Special Publication No. 8.

through A. The radius of the sphere is N, and its center is at H. Let A be the known station and B the unknown station. The angular distance of A from the pole is γ; the unknown distance of B is γ'; σ is the arc AB; α is the azimuth; and $\epsilon = 180^\circ - \alpha$

If γ' is computed by a direct solution of the spherical triangle ABP, the required precision can be reached only by the use of about ten-place logarithms. It is more convenient, and quite as accurate, for such short lines as occur in practice, to employ formulæ giving the *difference in latitude*, that is $\gamma - \gamma'$.

The formula for the direct solution of γ' in the spherical triangle is

$$\cos \gamma' = \cos \gamma \cos \sigma + \sin \gamma \sin \sigma \cos \epsilon. \tag{a}$$

Since γ' is a function of σ, its value may be expressed as a converging series by means of Maclaurin's formula (p. 408), giving

$$\gamma' = \gamma'_{\sigma=0} + \frac{d\gamma'}{d\sigma_{\sigma=0}} \cdot \sigma + \frac{1}{2} \cdot \frac{d^2\gamma'}{d\sigma^2{}_{\sigma=0}} \cdot \sigma^2 + \frac{1}{6} \frac{d^3\gamma'}{d\sigma^3{}_{\sigma=0}} \cdot \sigma^3 + \cdots. \tag{b}$$

To evaluate the three differential coefficients, differentiate Equa. (a) three times in succession, and in each resulting equation substitute $\sigma = 0$. The results of the first two differentiations are as follows:

$$-\sin \gamma' \frac{d\gamma'}{d\sigma} = -\cos \gamma \sin \sigma + \sin \gamma \cos \sigma \cos \epsilon, \tag{c}$$

$$\begin{aligned} -\sin \gamma' \frac{d^2\gamma'}{d\sigma^2} - \cos \gamma' \left(\frac{d\gamma'}{d\sigma}\right)^2 &= -\cos \gamma \cos \sigma - \sin \gamma \sin \sigma \cos \epsilon \\ &= -\cos \gamma', \text{ (by } (a)). \end{aligned} \tag{d}$$

Before differentiating a third time, (d) may be written

$$\tan \gamma' \frac{d^2\gamma'}{d\sigma^2} + \left(\frac{d\gamma'}{d\sigma}\right)^2 = 1. \tag{e}$$

Differentiating (e), we have

$$\tan \gamma' \frac{d^3\gamma'}{d\sigma^3} + \sec^2 \gamma' \cdot \frac{d\gamma'}{d\sigma} \cdot \frac{d^2\gamma'}{d\sigma^2} + 2 \frac{d\gamma'}{d\sigma} \cdot \frac{d^2\gamma'}{d\sigma^2} = 0. \tag{f}$$

When $\sigma = 0, \qquad \gamma' = \gamma,$
and (c) becomes

$$-\sin\gamma\,\frac{d\gamma'}{d\sigma} = \sin\gamma\cos\epsilon.$$

Therefore
$$\frac{d\gamma'}{d\sigma} = -\cos\epsilon. \tag{g}$$

(e) becomes

$$\tan\gamma\,\frac{d^2\gamma'}{d\sigma^2} + \cos^2\epsilon = 1.$$

Therefore
$$\frac{d^2\gamma'}{d\sigma^2} = \sin^2\epsilon\cot\gamma. \tag{h}$$

(f) becomes

$$\tan\gamma\,\frac{d^3\gamma'}{d\sigma^3} + \sec^2\gamma\,(-\cos\epsilon)(\sin^2\epsilon\cot\gamma) + 2\,(-\cos\epsilon)(\sin^2\epsilon\cot\gamma) = 0.$$

Therefore
$$\begin{aligned}\frac{d^3\gamma'}{d\sigma^3} &= \cos\epsilon\sin^2\epsilon\cot^2\gamma\,(2 + \sec^2\gamma)\\ &= (2\cot^2\gamma + \operatorname{cosec}^2\gamma)\sin^2\epsilon\cos\epsilon\\ &= (1 + 3\cot^2\gamma)\sin^2\epsilon\cos\epsilon \qquad (i)\end{aligned}$$

Substituting these results, (g), (h), and (i) in equation (b), Maclaurin's series, we obtain

$$\gamma' = \gamma - \sigma\cos\epsilon + \frac{\sigma^2}{2}\cdot\sin^2\epsilon\cot\gamma + \frac{\sigma^3}{6}(1 + 3\cot^2\gamma)\sin^2\epsilon\cos\epsilon + \cdots. \tag{j}$$

Changing to latitudes and azimuths by placing

$$\begin{aligned}\gamma' &= 90^\circ - \phi',\\ \gamma &= 90^\circ - \phi,\\ \epsilon &= 180^\circ - \alpha,\end{aligned}$$

Equation (j) becomes

$$\begin{aligned}\phi - \phi' = \sigma\cos\alpha &+ \frac{\sigma^2}{2}\sin^2\alpha\tan\phi\\ &\frac{\sigma^3}{6}(1 + 3\tan^2\phi)\sin^2\alpha\cos\alpha\ldots. \qquad (k)\end{aligned}$$

The difference in latitude being in radians on the sphere.

In order to transfer the coördinates of the triangulation points from the sphere to the spheroid, it should be noticed that if the radius of the sphere is N (the normal) and its center is at H (Fig. 99), and the polar axes of the sphere and spheroid coincide, then the parallels of latitude through A coincide, the spheroid being tangent to the sphere along this parallel; also, the latitude (ϕ) will be the same for both surfaces, and the distances and azimuths of AB on the two will differ by inappreciable quantities. We may therefore put $\sigma = \frac{s}{N}$, where s is the distance in linear units.*

Then (k) becomes

$$\phi - \phi' = \frac{s \cos \alpha}{N} + \frac{s^2}{2N^2} \sin^2 \alpha \tan \phi - \frac{s^3}{6N^3} \sin^2 \alpha \cos \alpha (1 + 3 \tan^2 \phi). \quad (l)$$

The difference in latitude should be measured, however, on a curve of radius R_m, since it is measured along the meridian $P'A$, Fig. 99. The linear difference in latitude is very nearly the same for the two surfaces, and the angular difference in latitude will vary inversely as the radii; that is,

$$s = (\phi - \phi') N = \Delta\phi'' R_M \text{ arc } 1''. \quad \text{(See Equa. [49].)} \quad (m)$$

Therefore $$\Delta\phi'' = (\phi - \phi') \frac{N}{R_M \text{ arc } 1''},$$

$\Delta\phi''$ being in seconds of arc on the meridian of the spheroid, and R_M the radius of curvature of the meridian at the *middle point* between the parallels through A and B. Therefore the difference in latitude (remembering that a positive value of $\cos \alpha$ corresponds with a decrease in latitude) is given by

$$-\Delta\phi'' = \frac{s \cos \alpha}{R_M \text{ arc } 1''} + \frac{s^2 \sin^2 \alpha \tan \phi}{2 N R_M \text{ arc } 1''} - \frac{s^3 \sin^2 \alpha \cos \alpha (1 + 3 \tan^2 \phi)}{6 N^2 R_M \text{ arc } 1''}. \quad (n)$$

The negative sign is introduced because when α is in the first

* The following more accurate expression is given by Clarke,

$$\frac{s}{N \sin 1''} + \frac{e^2 \sigma^3 \sin^2 1''}{6(1 - e^2)} \cos^2 \phi \cos^2 \alpha. \quad [63a]$$

or the fourth quadrant and $\cos\alpha$ is positive the difference in latitude is negative.

Since the middle latitude is not known at the beginning of the computation, it is more convenient first to take out the value of R_m for the known latitude of A, which will give an approximate difference in latitude, which we may call $\delta\phi''$, and then to change to that corresponding to R_M by multiplying by the inverse ratio of the radii.

$$\text{Since} \qquad \frac{\delta\phi''}{R_M} = \frac{\Delta\phi''}{R_m},$$

$$\Delta\phi'' = \delta\phi'' \frac{R_m}{R_M} = \delta\phi''\left(1 - \frac{R_M - R_m}{R_M}\right)$$

$$= \delta\phi''\left(1 - \frac{\Delta R_M}{R_M}\right)$$

$$= \delta\phi''\left(1 - \frac{dR_M}{R_M}\right) \text{approximately,}$$

in which $\delta\phi'' \frac{dR_M}{R_M}$ is a correction to be subtracted from the first value.

$$\text{From [43]} \qquad R_m = \frac{a(1 - e^2)}{(1 - e^2 \sin^2\phi)^{\frac{3}{2}}}.$$

$$\text{Therefore} \qquad dR_m = \frac{a(1 - e^2)\cdot 3e^2 \sin\phi \cos\phi \, d\phi}{(1 - e^2\sin^2\phi)^{\frac{5}{2}}}.$$

Since dR_m is the change from the starting point to the middle point, the differential $d\phi$ is taken as half the difference in latitude, $\delta\phi$; that is,

$$d\phi = \frac{\delta\phi'' \text{ arc } 1''}{2}$$

$$\text{Therefore} \qquad \delta\phi'' \frac{dR_m}{R_m} = \frac{3e^2 \sin\phi \cos\phi \text{ arc } 1''}{2(1 - e^2\sin^2\phi)}(\delta\phi'')^2$$

$$= D\cdot(\delta\phi'')^2. \qquad (o)$$

If we now put for brevity $\frac{1}{R_m \text{ arc } 1''} = B$, $\frac{\tan\phi}{2NR_m \text{ arc } 1''} = C$,

$\frac{s \cos \alpha}{R_m \text{ arc } 1''} = h$ (the first term in (n)), and $\frac{1 + 3 \tan^2 \phi}{6 N^2} = E$, then Equa. (n) becomes

$$-\Delta\phi'' = s \cdot B \cdot \cos \alpha + s^2 \cdot C \cdot \sin^2 \alpha + (\delta\phi'')^2 \cdot D - h \cdot s^2 \cdot E \cdot \sin^2 \alpha, \qquad [64]$$

the difference in latitude now being in seconds.

The new latitude is given by

$$\phi' = \phi + \Delta\phi''. \qquad [65]$$

The logarithms of the factors B, C, D, and E are given in Table XIV, p. 433, in metric units, for the Clarke spheroid of 1866.

The D term is inserted before the E term, because it is usually the larger. The E term may be omitted when log s is less than 4.23. . . . The D term may be omitted when log s is less than 2.31 . . . , and h^2 may be substituted for $(\delta\phi'')^2$ when log s is less than 4.93. . . . The fourth differential coefficient in the series

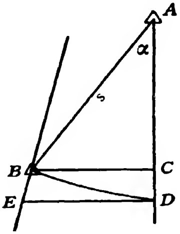

FIG. 100.

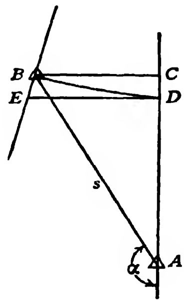

FIG. 100a.

may be neglected except for the very longest lines (see Coast Survey Report for 1894, p. 284).

Whenever the azimuth, α, is less than 90° or more than 270° (S W or S E) the B term is positive. Since the C and D terms contain squares they are always positive and are therefore added to the B term. The E term (h positive) is subtracted. When α

is between 90° and 270° (N W or N E) the B term is negative and the C and D terms are (numerically) subtracted from it, while the E term is added to it.

If a perpendicular is dropped from B to the meridian through A (Fig. 100) it is readily seen that the first term of the series (B term) is (very nearly) equal to the projection AC. This is the same as the "latitude" employed in plane surveying, but converted into seconds by the factor B. The difference in latitude between stations A and B is not AC, but the distance from A to the parallel of latitude through B. It is therefore necessary to add the distance CD, which is nearly equal to the offset at B from the tangent (DE) to the parallel of latitude. This is the C term. This term is proportional to the square of the distance out (BC or DE) and is the offset used in establishing a parallel of latitude in the surveys of the public lands.

120. Difference in Longitude.

The difference in longitude is such a small angle that we may obtain it with sufficient precision by a direct solution of the triangle PAB, Fig. 99, (but on a different auxiliary sphere), using 7-place logarithms.

Applying the law of sines,

$$\sin \Delta\lambda = \frac{\sin \sigma \sin \alpha}{\cos \phi'}$$

The sphere on which the points are projected is that whose radius is N' and whose center is at H' corresponding to point B.

As before, let $$\sigma = \frac{s}{N'}.$$

Therefore $$\sin \Delta\lambda = \sin \frac{s}{N'} \cdot \frac{\sin \alpha}{\cos \phi'}. \qquad (p)$$

In practice it is more convenient to solve the equation in the form

$$\Delta\lambda'' \text{ arc } 1'' = \frac{s}{N'} \cdot \sin \alpha \sec \phi',$$

and then to apply corrections for the difference between the arc

and sine; the equation should therefore be written

$$\Delta\lambda'' - \text{corr.}_{\log \Delta\lambda} = \frac{s}{N' \operatorname{arc} 1''} \cdot \sin \alpha \sec \phi' - \text{corr.}_{\log s},$$

since each side of the equation is too large by the difference between the arc and sine.

Placing $\frac{1}{N' \operatorname{arc} 1''} = A'$, the equation becomes

$$\Delta\lambda'' = A' \cdot s \cdot \sin \alpha \sec \phi' + \text{corr.}_{\log \Delta\lambda} - \text{corr.}_{\log s} \qquad [66]$$

in which the corrections are to be applied to the logarithms. Values of $\log A'$ will be found in Table XIV, p. 433.

In Art. 115, p 201, it was shown that

$$\log \frac{s}{R} - \log \sin \frac{s}{R} = \frac{Ms^2}{6R^2},$$

where s is the length of any line (great circle) on the surface.

If $\frac{s}{R}$ is an angle expressed in seconds, then the last equation becomes

$$\log \frac{s}{R} - \log \sin \frac{s}{R} = \frac{M\left(\frac{s''}{R}\right)^2 \operatorname{arc}^2 1'}{\ }$$

Taking logs of both members,

$$\log (\text{diff. of logs}) = \log\left(\frac{M \operatorname{arc}^2 1''}{6}\right) + 2 \log\left(\frac{s''}{R}\right).$$

Applying this formula first to $\Delta\lambda''$, we have

$$\log (\text{diff. of logs}) = 8.2308 + 2 \log \Delta\lambda''. \qquad (q)$$

Applying the formula to $\frac{s}{N'}$, and, observing that the second term is $2 \log \frac{s}{N' \operatorname{arc} 1''}$, we have

$$\log (\text{diff. of logs}) = 8.2308 + 2 \log s + 2 \log A' \qquad (r)$$

$$= 5.2488^* + 2 \log s. \qquad (s)$$

* Based on the value 8.5090 for $\log A'$.

This correction is to be subtracted because arc $\frac{s}{N'}$ is greater than $\sin\left(\frac{s}{N'}\right)$.

In Table XIII the corrections are tabulated to show the values of log s and log $\Delta\lambda''$ for the *same* log diff. The correction for log s is negative and that for log $\Delta\lambda''$ is positive. The algebraic sum of the two corrections is to be added to log $\Delta\lambda''$. The method of making these corrections is illustrated in the example on p. 220. The new longitude λ' is given by

$$\lambda' = \lambda + \Delta\lambda'' \qquad [67]$$

In the formula

$$\Delta\lambda'' = \frac{s}{N' \text{ arc } 1''} \sin\alpha \sec\phi'$$

it will be seen that $s \sin\alpha$ is the "departure" (of plane surveying or of navigation) expressed in linear units. This is reduced to seconds of arc of a great circle by the factor N' arc $1''$. This result is changed to seconds of arc of the parallel of latitude by the factor $\sec\phi'$.

In west longitudes a positive sign for the sine of α (α between $0°$ and $180°$) gives a positive sign to the difference in longitude. If α is between $180°$ and $360°$ the sine is $-$ and the longitude is decreased.

121. Forward and Back Azimuths.

Owing to the convergence of the meridians the forward and reverse azimuths of a line will not differ by exactly 180°, as in plane coördinates. The amount of this convergence is computed as follows

In the triangle *PAB*, Fig. 99, by Napier's analogies,

$$\tan \tfrac{1}{2}(A + B) = \cot \tfrac{1}{2}\Delta\lambda \cdot \frac{\cos \frac{1}{2}(\gamma' - \gamma)}{\cos \frac{1}{2}(\gamma' + \gamma)}.$$

Substituting, and noting that $A + B + \Delta\alpha = 180°$, and that

an increase in $\Delta\lambda$ causes a decrease in $\Delta\alpha$.

$$-\cot \tfrac{1}{2}\,\Delta\alpha = \cot \tfrac{1}{2}\,\Delta\lambda \frac{\cos \frac{1}{2}(\phi - \phi')}{\sin \frac{1}{2}(\phi + \phi')},$$

whence

$$-\tan \tfrac{1}{2}\,\Delta\alpha = \tan \tfrac{1}{2}\,\Delta\lambda \frac{\sin \frac{1}{2}(\phi + \phi')}{\cos \frac{1}{2}(\phi - \phi')}$$

$$= \tan \tfrac{1}{2}\,\Delta\lambda \frac{\sin \phi_m}{\cos \frac{\Delta\phi}{2}}.$$

Therefore

$$-\frac{\Delta\alpha}{2} = \tan^{-1}\left(\tan \frac{\Delta\lambda}{2} \cdot \frac{\sin \phi_m}{\cos \frac{\Delta\phi}{2}}\right),$$

Putting for $\frac{1}{2}\,\Delta\alpha$ the series

$$\left[\tan \tfrac{1}{2}\,\Delta\lambda \frac{\sin \phi_m}{\cos \frac{\Delta\phi}{2}}\right] - \tfrac{1}{3}\left[\tan \tfrac{1}{2}\,\Delta\lambda \frac{\sin \phi_m}{\cos \frac{\Delta\phi}{2}}\right]^3 + \cdots,$$

and for $\tan \frac{1}{2}\,\Delta\lambda$ the series

$$\tfrac{1}{2}\,\Delta\lambda + \frac{(\frac{1}{2}\,\Delta\lambda)^3}{3} + \cdots,$$

then

$$-\tfrac{1}{2}\,\Delta\alpha = \left[\left(\tfrac{1}{2}\Delta\lambda + \frac{\Delta\lambda^3}{24}\right)\frac{\sin \phi_m}{\cos \frac{\Delta\phi}{2}}\right] - \tfrac{1}{3}\left[\left(\tfrac{1}{2}\Delta\lambda + \frac{\Delta\lambda^3}{24}\right)\frac{\sin \phi_m}{\cos \frac{\Delta\phi}{2}}\right]^3 + \cdots$$

$$= \tfrac{1}{2}\,\Delta\lambda \frac{\sin \phi_m}{\cos \frac{\Delta\phi}{2}} + \frac{\Delta\lambda^3}{24} \cdot \frac{\sin \phi_m}{\cos \frac{\Delta\phi}{2}} - \frac{\Delta\lambda^3}{24} \cdot \frac{\sin^3 \phi_m}{\cos^3 \frac{\Delta\phi}{2}} + \cdots.$$

Multiplying by 2 and factoring out $\frac{\Delta\lambda^3}{24}$,

$$-\Delta\alpha = \Delta\lambda \frac{\sin \phi_m}{\cos \frac{1}{2}\,\Delta\phi} + \tfrac{1}{12}(\Delta\lambda)^3\left(\frac{\sin \phi_m}{\cos \frac{1}{2}\,\Delta\phi} - \frac{\sin^3 \phi_m}{\cos^3 \frac{1}{2}\,\Delta\phi}\right)$$

Placing $\cos \frac{1}{2}\,\Delta\phi = 1$ in the small term and reducing $\Delta\alpha$ and

$\Delta\lambda$ to seconds of arc,

$$-\Delta\alpha'' = \Delta\lambda'' \frac{\sin \phi_m}{\cos \frac{1}{2} \Delta\phi} + \tfrac{1}{12} (\Delta\lambda'')^3 \sin \phi_m \cos^2 \phi_m \operatorname{arc}^2 1''$$

$$= \Delta\lambda'' \sin \phi_m \sec \frac{\Delta\phi}{2} + (\Delta\lambda'')^3 \cdot F, \qquad [68]$$

in which F is an abbreviation for $\frac{1}{12} \sin \phi_m \cos^2 \phi_m \operatorname{arc}^2 1''$ and is given by its log in Table XIV*a*, p. 431. This F term amounts to only $0''.01$ when $\log \Delta\lambda'' = 3.36$. . . .

The back azimuth α' is given by

$$\alpha' = \alpha + \Delta\alpha'' + 180^\circ. \qquad [69]$$

It will be seen that the first term in the expression for $\Delta\alpha$ (omitting the factor $\sec \frac{\Delta\phi}{2}$, which is always nearly equal to unity) is the same as the formula used for calculating the convergence of the meridians, as when establishing township and section lines or when checking azimuths in a traverse.

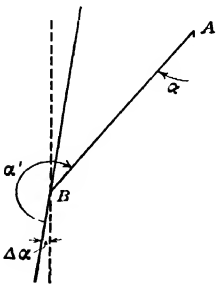

FIG. 101

From Fig. 101 it will be seen that if α is less than 180°, B is west of A, and $\Delta\alpha$ must be subtracted from α (and 180° added) to obtain α'.

In calculating the geodetic position of a point, the azimuth of the line to that point is to be found from the known azimuth of the fixed side of the triangle by using the corrected spherical angle, not the plane angle of the auxiliary triangle The computations of ϕ' and λ' may be verified by computing the position from two sides of the triangle and noting whether the same ϕ' and λ' are obtained from the two lines. The reverse azimuths are checked by noting whether their difference equals the spherical angle at the new station. In this manner the calculation of each triangle may be made to check itself.

122. Formulæ for Computation.

For convenience of reference the working formulæ are here brought together.

$$-\Delta\phi = s \cdot B \cdot \cos\alpha + s^2 \sin^2\alpha \cdot C + (\delta\phi'')^2 \cdot D - h \cdot s^2 \cdot \sin^2\alpha \cdot E,^{*} \quad [64]$$

$$\Delta\lambda = A' \cdot s \cdot \sin\alpha \sec\phi' \quad [66]$$

(or, $\log \Delta\lambda'' = \log s + \log \sin\alpha + \log A' + \log \sec\phi' + C_{\log \Delta\lambda} - C_{\log s}$),

$$-\Delta\alpha'' = \Delta\lambda'' \sin \tfrac{1}{2}(\phi + \phi') \sec \tfrac{1}{2}\Delta\phi + (\Delta\lambda'')^3 \cdot F, \quad [68]$$

in which

$$h = s \cdot \cos\alpha \cdot B,$$

$$-\delta\phi'' = s \cdot \cos\alpha \cdot B + s^2 \sin^2\alpha \cdot C - hs^2 \sin^2\alpha \cdot E.$$

The position of the new point and the reverse azimuth are then given by

$$\phi' = \phi + \Delta\phi'', \quad [65]$$

$$\lambda' = \lambda + \Delta\lambda'', \quad [67]$$

$$\alpha' = \alpha + \Delta\alpha'' + 180^\circ \quad [69]$$

The arrangement of the computation is illustrated by the following example. The two pages show the two computations of a position in the same triangle.

In the first page of the computation, the known station is *Waldo* and the position of *Bunker Hill* is to be found. Since the value of $\Delta\alpha$ depends upon $\Delta\lambda$ and $\Delta\lambda$ depends upon ϕ', the three parts of the solution must be carried out in the order indicated. In computing $\Delta\phi$, take out B, C, D, and E for the given latitude ϕ. The $(\delta\phi)$ used in the D term is usually taken as the algebraic sum of the first two terms of the series; if the E term is large, it should be included also. The h in the E term is the first (B) term alone. The algebraic signs of the functions of α are important and should be carefully attended to.

When computing $\Delta\lambda$, ϕ' is known and the factor $\log A'$ must

* The value of $-\Delta\phi$ may be made more accurate by the addition of the following term:

$$-\tfrac{1}{2} s^2 \cdot k \cdot E + \tfrac{3}{2} s^2 \cos^2\alpha \cdot k \cdot E + \tfrac{1}{3} s^2 \cdot \cos^2\alpha \sec^2\phi \cdot A^2 \cdot k \operatorname{arc}^2 1'',$$

in which $k = s^2 \cdot \sin^2\alpha \cdot C$.

α	Waldo to Meade's Ranch	255° 17′ 17″ 52
$\angle$	Meade's Ranch and Bunker Hill	86 20 54 50
α	Waldo to Bunker Hill	341 38 12 02
$\Delta\alpha$		+4 43 09
		180°
α'	Bunker Hill to Waldo	161 42 55 11
	Third angle	38 08 34 02

ϕ	39° 09′ 55″ 645	Waldo	λ	98° 49′ 50″ 128
$\Delta\phi$	−17 30 209	s = 34.407 64 meters	$\Delta\lambda$	−07 29 652
ϕ'	38 52 16 436	Bunker Hill	λ'	98 42 20 476

s	4 536 6549	s^2	9 07331	$(\delta\phi)^2$	6 0499	$-h$	3 0249 n
cos α	9 977 3018	sin² α	8 99674			s^2 sin² α	8 0700
B	8 510 9150	C	1 31553	D	2 3832	E	6 0871
h	3 024 8717		9 38558		8 4331		7 1820 n
1st term	1058″ 9409	3d term	+0 0271			$(\Delta\lambda)^3$	7 959
2d term	0 2429	4th term	−0 0015			F	7 872
	1059 1838		+0 0256				5 831
3rd and 4th		s	4 536 6540				
terms	+ 0256	sin α	9 498 3680 n	Arg		$\Delta\lambda$	2 652 877 n
$-\Delta\phi$	1059 2094	A'	8 509 1469	s	−21	sin ½ $(\phi + \phi')$	9 799 043
½ $(\phi + \phi')$	39° 01′ 06″.04	sec ϕ'	0 108 7088	$\Delta\lambda$	+03	sec ½ $(\Delta\phi)$	1
			2 652 8786 n	Corr.	−18		2 451 921 n
			18			$-\Delta\alpha$	−283″ 09
			2 652 8768 n				
		$\Delta\lambda$	−449″ 652				

be taken out for this new latitude ϕ', not for ϕ. The primes are inserted to call attention to this. To correct for the difference between the arc and the sine, enter Table XIII with log $\Delta\lambda$ and log s as arguments. The algebraic sum of the two values of "log diff." is the correction to be applied to log $\Delta\lambda$. The value of $\Delta\alpha$ is found last.

The values of ϕ' and λ' are checked by noting whether the same values are obtained from the two computations. The two reverse azimuths should differ by the spherical angle at the new station, which checks the computations of $\Delta\alpha$.

122a. Subsidiary Triangulation.

In calculating the positions of points in subsidiary triangulation the work may be somewhat simplified since it is unnecessary

α	Meade's Ranch to Waldo	75° 28′ 14″ 52
$\angle$	Bunker Hill and Waldo	55 30 33 73
α	Meade's Ranch to Bunker Hill	19 57 40 79
$\Delta\alpha$		−06 11 66
		180
α'	Bunker Hill to Meade's Ranch	199 51 29 13

ϕ	39° 13′ 26″ 686	Meade's Ranch	λ	98° 32′ 30″ 506
$\Delta\phi$	−21 10 250	s = 41,661 11 meters	$\Delta\lambda$	+09 49 969
ϕ'	38 52 16 436	Bunker Hill	λ'	98 42 20 475

s	4 619 7308	s^2	9 23946	$(\delta\phi)^2$	6 2076	$-h$	3 1037 n
$\cos\alpha$	9 973 0924	$\sin^2\alpha$	9 06649			$s^2\sin^2\alpha$	8 3060
B	8 510 9105	C	1 31644	D	2 3835	E	6 0882
h	3 103 7337		9 62239		8 5911		7 4979 n
1st term	+1269 795	3d term	+0 0390			$(\Delta\lambda)^2$	8 312
2d term	0 419	4th term	−0 0031			F	7 871
	+1270 214		+0 0359				6 183
3d and 4th		s	4 619 7308				
term	+0 036	$\sin\alpha$	9 533 2455	Arg		$\Delta\lambda$	2 770 830
$-\Delta\phi$	+1270 250	$1'$	8 509 1469	s	−31	$\sin\frac{1}{2}(\phi+\phi')$	9 799 317
		$\sec\phi'$	0 108 7088	$\Delta\lambda$	+06	$\sec\frac{1}{2}(\Delta\phi)$	2
$\frac{1}{2}(\phi+\phi')$	39° 02′ 51″ 56		2 770 8320	Corr.	−25		2 570 149
			−25				
			2 770 8295			$-\Delta\alpha$	371″ 66
		$\Delta\lambda$	+589″ 9694				

to make use of the smallest terms, such, for example, as the E term in the latitude equation, the corrections to the longitude, or the factor $\sec\frac{\Delta\phi}{2}$ and the F term in the azimuth equation.

The working formulæ would then take the form

$$-\Delta\phi'' = s \cdot B \cdot \cos\alpha + s^2 \cdot C \cdot \sin^2\alpha + (\delta\phi'')^2 D$$

$$\Delta\lambda'' = A' \cdot s \cdot \sin\alpha \sec\phi'$$

$$-\Delta\alpha'' = \Delta\lambda'' \sin\tfrac{1}{2}(\phi + \phi').$$

The computation would be carried out as shown on p. 222.

POSITION COMPUTATION, SUBSIDIARY TRIANGULATION.

		° ′ ″
α	Sand Point to La Salle	8 43 54 0
$\angle$	La Salle & Indianola	+44 46 17 3
α	2 Sand Point to 1 Indianola	53 30 11 3
$\Delta\alpha$		− 1 54 7
		180 00 00 0
α'	1 Indianola to 2 Sand Point	233 28 16 6
	Third angle of triangle	77 56 09 6

	° ′ ″			° ′ ″
ϕ	28 35 02 377	2 Sand Point	λ	96 26 59 604
$\Delta\phi$	− 2 36 805		$\Delta\lambda$	+ 3 59 900
ϕ'	28 32 25 572	1 Indianola	λ'	96 30 59 504

	° ′ ″	s	3 909 175	s^2	7 818		
$\frac{1}{2}(\phi + \phi')$	28 33 44	$\cos\alpha$	9 774 355	$\sin^2\alpha$	9 810	h^2	4 39
		B	8 511 666	C	1 142	D	2 32
	″	h	2 195 196		8 770		6 71
1st term	+156 7458				0589		0005
2d and 3d terms	+ 0594						
$-\Delta\phi$	+156 8052						

s	3 909 175		
$\sin\alpha$	9 905 196		
A'	8 509 391	$\Delta\lambda$	2 38003
$\sec\phi'$	0 056 268	$\sin\frac{1}{2}(\phi + \phi')$	9 67953
	2 380 030		2 05956
	″		″
$\Delta\lambda$	+239 900	$-\Delta\alpha$	+114 7

From Coast and Geodetic Survey Special Publication No. 8.

122b. Clarke's Formulæ for Computing Geodetic Positions.

The Puissant formulæ are sufficiently accurate for lines up to about 70 or 80 miles in length and these usually satisfy all the requirements of triangulation. In a few instances, however, it has been necessary to employ more accurate formulæ for computing the positions. The following solution of the problem was given by Clarke in the Report of the *Ordnance Survey of Great Britain* 1858, and in his *Geodesy*; these formulæ were used

in the calculation of the Davidson Quadrilaterals in California and Nevada.

In this solution the triangle formed by joining the two stations and the pole, Fig. 99, is first solved directly in order to obtain (simultaneously) the angle at B and the difference in longitude. The calculation is ordinarily made with 10-place logarithms. The interior angles, rather than the azimuths are employed in the solution. The difference in latitude is found last. Using the same notation as before, except for the azimuths, we have

$$\sigma'' = \frac{s}{N \sin 1''} + \frac{e^2\sigma^3 \sin^2 1''}{6\,(1 - e^2)} \cos^2 \phi \cos^2 \alpha \qquad [69a]$$

$$\zeta'' = \frac{e^2\sigma^2 \sin 1''}{4\,(1 - e^2)} \cos^2 \phi \sin 2\alpha \qquad [69b]$$

$$\tan \tfrac{1}{2}\,(\alpha' + \zeta + \Delta\lambda) = \frac{\cos \frac{1}{2}\,(\gamma - \sigma)}{\cos \frac{1}{2}\,(\gamma + \sigma)} \cot \frac{\alpha}{2} \qquad [69c]$$

$$\tan \tfrac{1}{2}\,(\alpha' + \zeta - \Delta\lambda) = \frac{\sin \frac{1}{2}\,(\gamma - \sigma)}{\sin \frac{1}{2}\,(\gamma + \sigma)} \cot \frac{\alpha}{2} \qquad [69d]$$

$$\phi' - \phi = \frac{s}{R_M \sin 1''} \cdot \frac{\sin \frac{1}{2}\,(\alpha' + \zeta - \alpha)}{\sin \frac{1}{2}\,(\alpha' + \zeta + \alpha)} \left[1 + \frac{\sigma^2 \sin^2 1''}{12} \cdot \cos^2 \tfrac{1}{2}\,(\alpha' - \alpha) \right]. \qquad [69e]$$

In these formulæ ζ is the small angle (at B) between the two plane curves, α and $\alpha' + \zeta$ are the interior angles of the triangle PAB, Fig. 99. The computations are necessarily carried out in the order given above. Positions obtained by these formulæ are reliable up to about 200 miles.

122c. Dalby's Theorem.

In deriving formula [68] and also formulæ [69*c*] and [69*d*] it was assumed that the spherical formula (Napier's analogies) would give the difference between the forward and back azimuths on the surface of the spheroid. Dalby showed that the error in this assumption is negligible, and it is usually known as Dalby's Theorem. Helmert (Höheren Geodäsie, Vol. I, p. 150)

shows that the error may be expressed in the form

$$a_{BA} - a_{AB} = a'_{BA} - a'_{AB} + \frac{e^4}{4 \text{ arc } 1''} \sin\left(\frac{s}{N}\right) \sin^2 \Delta\phi \sin\phi \cos^4\phi \cdots,$$

and that this small term is ordinarily less than one-thousandth of a second. The spheroidal azimuths are indicated by a_{AB} and a_{BA}, the spherical azimuths being indicated by the primed letters.

Clarke, (Geodesy, p. 106) shows that if α and α' are the interior angles (spheroidal) and β and β' are the corresponding spherical angles, then,

$$\alpha + \alpha' = \beta + \beta' + \frac{e^4}{4 \text{ arc } 1''}\left(\frac{k}{a}\right)^3 \sin\alpha \cos^2\alpha \sin\phi \cos^3\phi \cdots,$$

k being the chord distance. This is nearly the same as the preceding. The small term is less than a ten-thousandth of a second for the longest lines which can be sighted.

From these equations we infer (1) that the difference in azimuth at two stations may be calculated as spherical without sensible error, and (2) the spheroidal excess equals the spherical excess of a spherical triangle having for its vertices the pole and two points having the same latitudes and longitudes as the stations in question.

123. The Inverse Problem.

Not infrequently it is required to find the distance and mutual azimuths between two stations whose latitudes and longitudes are known.

If we place $x = s \sin\alpha$ and $y = s \cos\alpha$, then, from Equa. [66] and [64], we have

$$x = \frac{\Delta\lambda'' \cos\phi'}{A'} \qquad [70]$$

and $$y = -\frac{1}{B}[\Delta\phi'' + Cx^2 + D(\delta\phi'')^2 + E(\Delta\phi'')x^2], \qquad [71]$$

from which $$\tan\alpha = \frac{x}{y} = \frac{\Delta\lambda \cos\phi' B}{A' \cdot h} \qquad [72]$$

and $$\left.\begin{aligned} s &= y \sec\alpha \\ &= x \operatorname{cosec}\alpha. \end{aligned}\right\} \qquad [73]$$

The inverse solution may be worked out on the same printed form that is used for the direct solution, but the order of procedure is modified as follows: First, compute x by Equa. [70], then the C, D (and E) terms in Equa. [71], obtaining finally y. The azimuth is then found through its tangent; s may be calculated from the x term or from the y term, whichever is more accurate. The calculation of $\Delta\alpha$ gives α'.

In the following example of a solution of the *inverse* problem, the two latitudes and the two longitudes are known and s, α, and α' are to be found. The only difference between this form and the preceding one is the calculation of tan α and s in the lower right-hand corner.

POSITION COMPUTATION, SECONDARY TRIANGULATION.

INVERSE SOLUTION

α	to	° ′ ″
$\angle$	&	
		+
α	2 Sand Point to 1 Indianola	53 30 11 5
$\Delta\alpha$		− 1 54 7
		180 00 00 00
α'	1 Indianola to 2 Sand Point	233 28 16 8
	Third angle of triangle	

	° ′ ″			° ′ ″
ϕ	28 35 02 377	2 Sand Point	λ	96 26 59 604
$\Delta\phi$	− 2 36 805		$\Delta\lambda$	+ 3 59 900
ϕ'	28 32 25 572	1 Indianola	λ'	96 30 59 504

	° ′ ″						
$\frac{1}{2}(\phi + \phi')$	28 33 44	s $\cos\alpha$	3 683 529	s^2 $\sin^2\alpha$	7 6287	h^2	4 39
	″	B	8 511 666	C	1 1110	D	2 32
1st term	+156 7455	h	2 195 195		8 7706		6 71
2d and 3d terms	+ 0595				0590		0005
$-\Delta\phi$	+156 8050						

s $\sin\alpha$	3 814 371		
A'	8 509 391	$\Delta\lambda$	2 380 03
$\sec\phi'$	0 055 268	$\sin\frac{1}{2}(\phi + \phi')$	9 679 53
	2 380 030		2 059 56
	″		″
$\Delta\lambda$	+239 900	$-\Delta\alpha$	+114 7

$s \sin\alpha = 3\ 814\ 371$

$s \cos\alpha = 3\ 683\ 529$

$\tan\alpha = 0\ 130\ 842$

° ′ ″

$\alpha = 53\ 30\ 11\ 5$

$\cos\alpha = 9\ 774\ 355$

$s = 3\ 909\ 174$

(From Coast and Geodetic Survey Special Publication No. 8.)

123a. Tabulation of Data Derived from Triangulation.

In tabulating the geodetic positions, distances, azimuths, etc. for publication the following form is frequently used. The column "seconds in meters" is useful when plotting the position on a sheet showing the minutes of latitude and longitude.

Station	Latitude and longitude.	Secs. in meters.	Azimuth	Back Azimuth	To station	Distance.	Logarithm of distance
	° ′ ″		° ′ ″	° ′ ″			
Unkonoonuc	42 58 59.864	1847 4	249 31 12 62	70 12 57 35	Agamenticus	77686 65	4 8903464
(N.H.) 1860	71 35 19.380	439 9	196 35 18.88	16 44 18 82	Gunstock	61992 11	4 7923364
Monadnock	42 51 41.174	1270 6	219 13 05 04	39 43 25 22	Gunstock	94468 57	4 9752873
(N.H.) 1860	72 06 30 776	698 6	252 07 57 59	72 23 12 03	Unkonoonuc	44548 68	4 6488348

124. Location of Boundaries.

Whenever it becomes necessary to establish on the ground a boundary line between two states or countries, the length of the lines and the accuracy demanded usually make it necessary to employ geodetic methods. A boundary may consist of a meridian arc, a parallel of latitude, or a great circle inclined to the meridian; or it may be a combination of these.

125. Location of Meridian.

If a boundary is a meridian arc the longitude of which is fixed by law, it is first necessary to assume approximate positions for the terminal points, and then to determine the longitude of these by direct observations. These points are then corrected in position. After the terminals have been established on the ground, the line may be run from one to the other as a random line, to be subsequently corrected if necessary. Observations on Polaris for azimuth will show the direction of the meridian. The line is then run out by backsighting and foresighting. If necessary, the direction of the meridian may be determined at intermediate points. When the second point is reached, the error in the running of the line becomes known, and the random line may be set over or re-run in the usual manner. If the boundary is long, the intermediate points may be found by triangulation in-

stead of by direct measurement. In any case triangulation will furnish a valuable check.

126. Location of Parallel of Latitude.

In order to establish a parallel of latitude on the ground, it is necessary to assume a point as nearly as may be on the desired parallel. The exact position of this assumed point is then determined by Talcott's method, and the station moved, if necessary, to the correct position. If the difference between the observed and the desired latitude is $\Delta\phi$, the sea-level distance which the station must be moved is $s' = R_m \Delta\phi'' \cdot \text{arc } 1''$.

At higher elevations s' should be increased in proportion to the distance from the center of the earth (Equa. [6]). If the error in position proves to be large, it may be advisable to make another determination of the latitude, in order to avoid the effect of station errors. (See Art. 83, p. 151.)

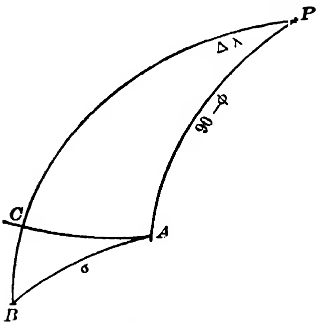

FIG. 102.

The next step is to determine the azimuth of a reference mark, by observation on Polaris, and to establish the direction of a great circle at right angles to the meridian (prime vertical). Points on the parallel of latitude are then determined by measuring offsets from the prime vertical as a reference line.

In Fig. 102 we have, in the triangle PAB,

$$PA = 90^\circ - \phi,$$

$$A = 90^\circ,$$

and
$$\tan \sigma = \tan \Delta\lambda \cos \phi,$$

or
$$\sigma = \tan^{-1} (\tan \Delta\lambda \cos \phi)$$

Expanding σ by the formula for $\tan^{-1} x$, p. 408, and also

tan $\Delta\lambda$ in terms of $\Delta\lambda$ by the formula for tan x, p. 408, we have

$$\sigma = \Delta\lambda \cos\phi + \tfrac{1}{3}(\Delta\lambda\cos\phi)^3 \tan^2\phi,$$

or

$$s = \sigma N = N\Delta\lambda'' \cdot \cos\phi \cdot \text{arc } 1'' + \tfrac{1}{3} N (\Delta\lambda'' \cos\phi \cdot \text{arc } 1'')^3 \tan^2\phi, \qquad [74]$$

which gives the distance AB corresponding to any difference in longitude $\Delta\lambda''$.

If in Equa. (n), p. 211, we place $\alpha = 90°$,

$$-\Delta\phi'' = \frac{s^2 \tan\phi}{2\, N R_M \text{ arc } 1'}$$

The offset P from the prime vertical (tangent) for any distance s from the initial point is

$$P = \Delta\phi'' R_M \text{ arc } 1'' = \frac{s^2 \tan\phi}{2\, N} \qquad [75]$$

Since P varies as s^2, the offsets for equidistant intervals along the line may be readily calculated. The direction of the pole from any point (x) on AB is given by

$$PxA = 90° + \Delta\alpha,$$

in which it is sufficiently accurate to take

$$-\Delta\alpha'' = \Delta\lambda'' \sin\phi_m. \qquad [76]$$

Since the numerical value of $\Delta\alpha''$ increases directly as $\Delta\lambda''$, it will be sufficient to take the increments of $\Delta\alpha''$ as proportional to s.

If the arc of the parallel is a long one, it is advisable to break it into sections, and to establish a new point at the beginning of each section by direct latitude observation.

(See *United States Northern Boundary Survey*, Washington, 1878.)

127. Location of Arcs of Great Circles.

The general method of laying out arcs not coincident with the meridian is that of determining astronomically the latitudes and longitudes of the terminal points, and then running a random line between them. The direction and distance between the

terminals may be found by Formulæ [70] to [73] for the inverse solution of the geodetic problem. The azimuth is determined by observation at intermediate points. The error of the random line is corrected in the usual way. For long arcs triangulation would be substituted for direct measurement.

(See Appendix 3, Coast Survey Report for 1900, "The Oblique Boundary Line between California and Nevada.")

128. Plane Coördinate Systems.

When all the points to be located in a survey are comprised within a relatively small area, such as a city or a metropolitan district, the calculations are greatly simplified by the use of plane coördinates. Many of the large cities, New York, Cincinnati, Rochester, Atlanta, Boston, Baltimore, and others, have adopted such a system. If there are reliable triangulation points already established within the area, these will naturally be used as a basis for the new survey, or at any rate to check the new triangulation.

In establishing a system of plane coördinates it is necessary to decide first upon the positions of the coördinate axes. These will naturally be a meridian and a great circle at right angles to it; or, more properly speaking, they will be straight lines tangent to these two circles at their point of intersection, all points being supposed to lie in the plane defined by these two lines. The origin of the system must be defined in terms of the coördinates of some specified point of the survey (geodetic datum, p. 206). Unless this is done, the origin will not be the same when derived from different points, and ambiguity will exist regarding the true position of the origin. The origin may be taken as coincident with the selected triangulation point, as in the case of the survey of Boston, Massachusetts, and Baltimore, Maryland; or it may be the intersection of a selected meridian and parallel as derived from the assigned latitude and longitude of some station. In Springfield, Massachusetts, for example, the origin is the intersection of the $42° 04'$ parallel and the $72° 28'$ meridian, as determined by the published latitude and longitude of the United States Armory flagpole. The direction of the meridian must be

defined as making a certain angle with a specified line of the survey, preferably one which passes through the fundamental point.

The point at which the plane is tangent to the spheroid must not be confused with the (o, o) point of the system. The former should be within the area surveyed, preferably at its center, in order to avoid large spherical errors. The latter may be taken at any convenient distance outside the area by assigning to the tangent point large values of x and y, in order to avoid negative values in the coördinates of the survey points. The tangent point is on the sphere as well as on the plane; the (o, o) point is not necessarily on the sphere. In case the area is large it is sometimes advisable to use more than one tangent point. This was done in the survey of the city of New York.

129. Calculation of Plane Coördinates from Latitude and Longitude.

In calculating the plane coördinates of a point, we may apply Formulæ [70] to [73] for the inverse solution of the geodetic

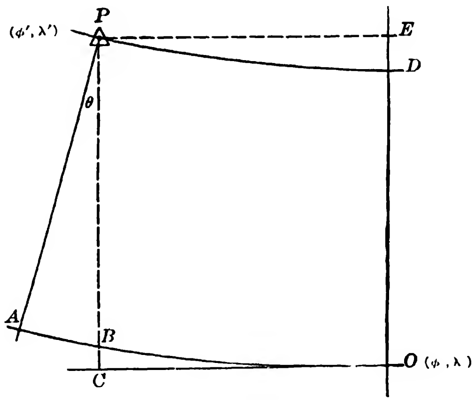

FIG. 103.

problem, one of the points being the origin (tangent point) whose coördinates are ϕ and λ, and the other the triangulation point the coördinates of which are ϕ' and λ'. The x and y there given are

the plane coördinates desired. If the coördinates of many points are to be transformed, it will prove to be more convenient to use specially prepared auxiliary tables and to modify the calculations as follows.

In Fig. 103 P is the triangulation point whose latitude and longitude are known, and whose coördinates x and y with reference to the origin O are desired. For such distances as are likely to occur in a plane system it may be assumed that $PE = PD$; that is, x equals the length of the arc of the parallel PD. The ordinate $y = PC$ may be taken as PA (the difference in latitude) plus BC* (the offset from great circle to parallel). From Formula [70],

$$x = PD = \Delta\lambda'' \cdot \frac{\cos \phi'}{A'} \cdot \qquad [77]$$

If x is to be expressed in feet,

$$\begin{aligned} x &= \Delta\lambda'' \cdot \frac{\cos \phi'}{A'} \times 3.2808\tfrac{1}{3} \qquad [78] \\ &= \Delta\lambda'' \times H. \end{aligned}$$

Values of $\log H$ for every 30″ of latitude from $\phi = 42° \ 10'$ to $\phi = 42° \ 30'$ are given in Table A. Proportional parts for the seconds are given at the right.

Since the factors A' and $\cos \phi'$ appear in Equa. [78] we should use latitude ϕ' when taking out the factor H.

* If P is south of the origin, the offset must be subtracted.

TABLE A. VALUES OF LOG $\frac{\cos \phi'}{A'}$ + 0.515 9842*

Distance west of origin in feet $= x = \Delta\lambda'' \times H$

Lat. ϕ'. ° ′ ″	Log H.	Lat. ϕ'. ° ′ ″	Log H.
42 10	1.876 8536	42 20	1 875 7103
30	7966	30	6530
11	7396	21	5957
30	6825	30	5383
12	6255	22	4809
30	5684	30	4235
13	5114	23	3661
30	4543	30	3086
14	3971	24	2512
30	3400	30	1937
15	2828	25	1362
30	2256	30	0787
16	1684	26	1.875 0212
30	1112	30	1 874 9636
17	1 876 0541	27	9061
30	1 875 9968	30	8485
18	9396	28	7910
30	8823	30	7334
19	8250	29	6757
30	7677	30	6181
20	1.875 7103	30	1 874 5604

P P ″	570	572	574	576
1	19	19	19	19
2	38	38	38	38
3	57	57	57	58
4	76	76	77	77
5	95	95	96	96
6	114	114	115	115
7	133	134	134	134
8	152	153	153	154
9	171	172	172	173
10	190	191	191	192
11	209	210	210	211
12	228	229	230	230
13	247	248	249	250
14	266	267	268	269
15	285	286	287	288
16	304	305	306	307
17	323	324	325	326
18	342	343	344	346
19	361	362	364	365
20	380	381	383	384
21	399	400	402	403
22	418	419	421	422
23	437	439	440	442
24	456	458	459	461
25	475	477	478	480
26	494	496	497	499
27	513	515	517	518
28	532	534	536	538
29	551	553	555	557
30	570	572	574	576

* This is the form adopted by the city of Springfield, Mass., for its coördinate system.

Note: More complete tables, extending from lat. 24° to lat. 49° (10″ intervals) will be found in U S. Coast and Geodetic Survey Special Publ No. 71. (1921).

TABLE B. VALUES OF 0.515 9842 − log B

$$\text{Dist. N. of Origin in Feet} = \Delta\phi'' \times K + x^2 \frac{\tan\phi}{2N}$$

$$\text{Dist. S. of Origin in feet} = \Delta\phi'' \times K - x^2 \frac{\tan\phi}{2N}$$

Lat.			Log. K.	Lat.			Log. K.	P. P., Diff. 1′ = 12 8.			
°	′	″		°	′	″		″		″	
42	10		2 005 2981	42	20		2 005 3109	1	0	22	5
		30	2988			30	3116	2	0	23	5
	11		2994		21		3122	3	1	24	5
		30	3000			30	3129	4	1	25	5
	12		3006		22		3135	5	1	26	6
		30	3013			30	3141	6	1	27	6
	13		3019		23		3147	7	1	28	6
		30	3026			30	3154	8	2	29	6
	14		3032		24		3160	9	2		
		30	3039			30	3167	10	2		
	15		3045		25		3173	11	2		
		30	3052			30	3180	12	3		
	16		3058		26		3186	13	3		
		30	3064			30	3193	14	3		
	17		3070		27		3199	15	3		
		30	3077			30	3205	16	3		
	18		3083		28		3211	17	4		
		30	3090			30	3218	18	4		
	19		3096		29		3224	19	4		
		30	3103			30	3231	20	4		
	20		2 005 3109		30		2.005 3237	21	4		

The difference in latitude PA is converted into feet by multiplying $\Delta\phi''$ by $\frac{3.2808\frac{1}{3}}{B}$ ($= K$). (See Table B.) Use the *middle* latitude when taking out this factor K (D term not used).

The offset BC (Formula [75]) $= \frac{\tan\phi}{2N} \times x^2$.* [79]

The factor $\frac{\tan\phi}{2N}$ ($= L$), in feet, may be taken from Table C which was calculated by the formula

$$\log\frac{\tan\phi}{2N} = \log C - \log B - \log 3.2808\tfrac{1}{3}. \qquad [80]$$

* For another method of calculating this offset, see an article entitled "A Method of Transforming Latitude and Longitude into Plane Coördinates," by Sturgis H Thorndike, *Journal Boston Society Civil Engineers*, Vol. 3, No. 7, September, 1916.

TABLE C. VALUES OF LOG $\frac{\tan\phi}{2N}$ (ft.) $= \log C - \log B - 0.515\ 9842$

Offset from parallel $= \log L + 2 \log x$

Lat.	Log. L.	Lat.	Log. L.	P. Diff. 1′ = 25 4.			
° ′ ″		° ′ ″		″		″	
42 10	2 33 460	42 20	2 33 714	1	0	24	10
30	473	30	727	2	1	25	11
11	486	21	739	3	1	26	11
30	499	30	752	4	2	27	11
12	512	22	765	5	2	28	12
30	525	30	778	6	3	29	12
13	537	23	790	7	3		
30	550	30	803	8	4		
14	562	24	815	9	4		
30	575	30	828	10	4		
15	587	25	840	11	5		
30	600	30	853	12	5		
16	612	26	865	13	6		
30	625	30	878	14	6		
17	638	27	892	15	6		
30	651	30	905	16	7		
18	663	28	917	17	7		
30	676	30	930	18	8		
19	689	29	942	19	8		
30	702	30	955	20	8		
	2 33 714	30	2.33 967	21	9		
				22	9		
				23	10		

Example. As an illustration of how this method would be applied, let us suppose that it is desired to compute the plane coordinates of △ *Powderhorn* in a system whose origin is the dome of the State House, Boston, Massachusetts. We first compute $\Delta\phi''$ and $\Delta\lambda''$ and then apply formulæ [78], [79] and [80] as shown.

	Lat.	Long.
Powderhorn	42° 24′ 04″.683	71° 01′ 52″.006
State House	42 21 29 .596	71 03 51 .040
	2′ 35″.087	1′ 59″.034
	$\Delta\phi'' = 155''.087$	$\Delta\lambda'' = 119''.034$

log x^2 7.90183 | log $\Delta\phi''$ = 2.190 5754 | log $\Delta\lambda''$ 2.075 6710

log L 2.33752 | log K = 2.005 3145 | log H 1.875 2422

log = 0.23935 | log 4.195 8899 | log x = 3 950 9132

Offset = 1.7352 ft. | 15699.65 ft. | x = 8931.27 ft. East of State House.

1.74

y = 15701.39 ft. North of State House

If it is preferred to make the conversion from $\Delta\lambda$ to x always on the same parallel of latitude, that of the origin, a table may be calculated, giving the length of each minute (1′ to 10′) and each

second (1″ to 60″) of arc on this parallel; the difference in longitude may be taken out, by parts, from this table. If this is done, however, it is necessary to make allowance for the convergence of the meridians between the two parallels by solving for the distance $AB = y \sin \theta$ (Fig. 103). The convergence $\theta = \Delta\lambda'' \sin \phi_m$ and its sine may be tabulated for different values of $\Delta\lambda$ and ϕ_m. If the triangulation point is north of the origin, AB is to be subtracted; if south, it is to be added.

The plane coördinates computed in this manner are based upon sea-level distances, since all the triangulation was reduced to sea-level through the reduction of its measured base-lines. If the mean elevation of the district is high the positions of the triangulation points as given by these x, y, coördinates will not be consistent with those obtained directly by field measurement. If the line OP (Fig. 103) has an elevation of 1000 feet the computed position of P will be nearer to O by about $\frac{1}{20,000}$th part of the distance OP than it would if measured directly on the surface of the ground. In order to avoid this difficulty, and also the inconvenience of reducing all measurements to sea-level, it is customary to correct the coördinates of such triangulation stations by increasing the distances in proportion to the mean height of the district above sea-level. (Equa. [6].) If this is done there need be no further reduction to sea-level because all distances will now be on the same surface of reference. The position of the origin, however, must still be considered as its sea-level position in order that this survey may have its correct location with reference to other surveys.

130. Errors of a Plane System.

In order to investigate the errors of a plane coördinate system like the preceding, let us assume that a line starts from the origin O, Fig. 104, in an azimuth α, and follows the surface of a sphere of radius $\sqrt{R_m N}$ (for latitude ϕ) for a distance s meters, to point A; and that another line OA', having the same azimuth and length, lies in the plane which is tangent to the sphere at O. The point A' in the plane then represents the point A on the sphere as de-

termined by a direct measurement from the origin. The defects of the plane system as a means of representing points on a sphere will be shown by the error in reproducing point A' by following different routes, such, for example, as traversing due north and then due west on the sphere, or due west and then due north.

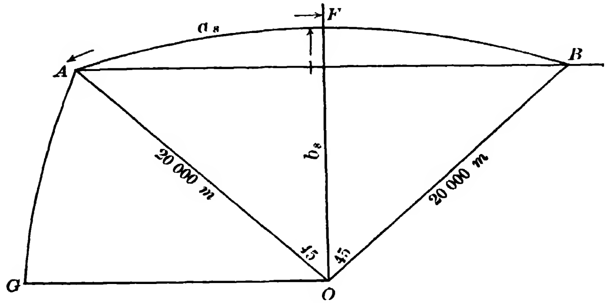

FIG 104.

If a perpendicular AF (an arc of a great circle) be let fall from A (Fig. 104) to the meridian through O, its length will be determined by

$$\sin \frac{}{R} = \sin \frac{}{R} \quad \sin \alpha,$$

where a is the perpendicular distance in meters and R is the radius of the sphere.

For the corresponding distance on the plane,

$$a = s \cdot \sin \alpha.$$

Distinguishing the plane and spherical values of a by subscripts, p and s, the difference in length may be found as follows:

$$a_p - a_s = s \sin \alpha - R \sin^{-1}\left(\sin \alpha \sin \frac{s}{R}\right)$$

$$= s \cdot \sin \alpha - R\left[\sin \alpha \left(\frac{s}{R} - \frac{s^3}{6R^3}\right) + \frac{\sin^3 \alpha}{6}\left(\frac{s}{R} - \frac{s^3}{6R^3}\right)^3\right]$$

$$= s \cdot \sin \alpha - \frac{Rs \sin \alpha}{R} + \frac{s^3}{6R^2} \sin \alpha - \frac{s^3}{6R^2} \sin^3 \alpha + \cdots$$

$$\frac{s^3}{6R^2} \sin \alpha \cos^2 \alpha +$$

Assuming that $\phi = 40°$, $\alpha =$ N 45° W, and $s = 20{,}000$ meters (about 12 miles), then $a_p - a_s = 0^m.0116$ If another such line were to extend $20{,}000^m$, N 45° E, to B, the terminal points A and B would then be $0^m.0232$ farther apart if calculated on a plane than if calculated on the sphere.*

If the survey proceeds from O northward to the point F, where the great circle from A, perpendicular to the meridian, intersects that meridian, and then westward along this great circle to A, the point A would be reached without error, if the measurements were perfect. The point computed on the plane would not agree, however, with A' as already established. The excess of the spherical distance b_s, along the meridian to the foot of the perpendicular F, over the plane distance b_p is found as follows:

In the spherical right triangle,

$$\tan\frac{b}{R} = \tan\frac{s}{R}\cos\alpha.$$

Then

$$\begin{aligned} b_s - b_p &= R\tan^{-1}\left(\tan\frac{s}{R}\cos\alpha\right) - s\cos\alpha \\ &= \frac{s^3\cos\alpha\sin^2\alpha}{3R^2}. \end{aligned}$$

Assuming the same data as before, we find that in order to reach A, on the sphere, we must run N 14142.15886 meters and then W 14142.12400 meters. Since in this case $s\sin\alpha = s\cos\alpha = 14142.13563^m$, such a traverse, when computed on the plane, gives a point $0^m.02323$ N and $0^m.01163$ E of point A'. A similar traverse running west to point G (Fig. 104) and then north to A would give a point $0^m.01163$ S and $0^m.02323$ W of point A'. The relative positions are shown (actual size) in Fig. 105.

The maximum discrepancy in the traverse is then about $0^m.05$, or about two inches. This would appear as an error of closure of the traverse $OFAGO$ even if there were no error whatsoever in the measurements themselves.

* This does not refer to the chord-distance AB, but to the distance on the spherical surface.

The difference in length between an arc of the parallel and an arc of the great circle is found as follows: In Fig. 106, $\frac{1}{2} AB = r \sin \frac{\Delta\lambda}{2} = R \sin \frac{\theta}{2}$. Replacing the sines by their series in terms of the arcs, $r\left(\frac{\Delta\lambda}{2} - \frac{\Delta\lambda^3}{48}\right) = R\left(\frac{\theta}{2} - \frac{\theta^3}{48}\right)$. The difference between $r\,\Delta\lambda$, the arc of the parallel, and $R\theta$, the arc of the great circle, is

$$r\Delta\lambda - R\theta = r\,\frac{\Delta\lambda^3}{24} - R \cdot \frac{\theta^3}{24}$$

$$= R\cos\phi\,\frac{\Delta\lambda^3}{24} - R \cdot \frac{\Delta\lambda^3 \cos^3\phi}{24} \text{ (approx.)}$$

since $\theta = \Delta\lambda \cos\phi$, nearly.

Therefore $r\Delta\lambda - R\theta = \frac{1}{24} R \cos\phi\, \Delta\lambda^3 (1 - \cos^2\phi)$
$= \frac{1}{24} R\,(\Delta\lambda'')^3 \cdot \text{arc}^3\, 1'' \cos\phi \sin^2\phi.$

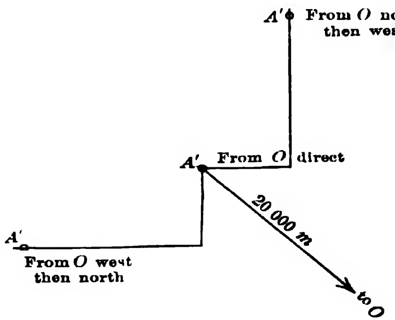

FIG. 105.

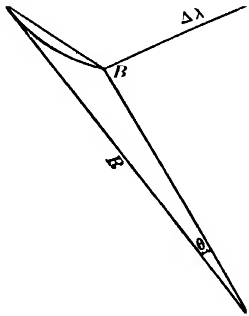

FIG. 106.

In order to compare this with the previous examples, we must put $\Delta\lambda'' = 1192''.4$, which corresponds to the distance between A and B. The error $r\Delta\lambda - R\theta$ is found to be $0^m.0186$ for the total arc, or $0^m.0093$ for the half arc. The difference between the length of the parallel and the x coördinate is therefore $0^m.0116 - 0^m.0093 = 0^m.0023$.

These results indicate that a plane system may be extended

over an area twelve miles in radius without involving errors of computation as great as the errors of measurement, and also that the formulæ given may be used whenever it is safe to use plane coördinates.

The error resulting from the use of plane coördinates when carried to various distances from the origin is shown in the following table, taken from U. S. Coast and Geodetic Survey, Special Publication No. 71.

Distance from origin		Accuracy of plane coördinates
[illegible]meters.	Miles	
30	18.6	1 part in 100 000
40	24.9	1 " 50 000
64	40	1 " 20 000
90	55.9	1 " 10 000
128	79.5	1 " 5 000

131. Traverses.

For methods of executing first order traverses and adjusting them to established triangulation, reference is made to Special Publication No. 137, U. S. Coast and Geodetic Survey.

PROBLEMS

Problem 1. Calculate the latitude and longitude of point A, Problem 3, Chapter VI, from both lines, and the back azimuths AB and AC.

Problem 2. Calculate the latitude and longitude of point E, Problem 4, Chapter VI, and the back azimuths EL and EN.

Problem 3. Calculate the position of Sand Hill in Problem 5, Chapter VI.

Problem 4. What will be the error of closure of a survey which follows the circumference of a circle whose radius is 20,000 meters (on the earth's surface) if the survey is calculated as though it were on a plane, the latitude of the center being 40° N. and the measurements being exact?

Problem 5. What error will be caused by dropping the small term in formula [63*a*] (footnote) if the distance is 200 miles and $\phi = 45°$, and $\alpha = 0°$?

Problem 6. Compute the position of B by formulas [64] to [69] and by formulæ [69*a*] to [69*e*].

$$\begin{aligned} \text{The latitude of } A &= 41°\ 01'\ 17''.240 \text{ N} \\ \text{Longitude of } A &= 114°\ 04'\ 36''.286 \text{ W} \\ \text{Distance } AB &= 237\ 770.57^m \text{ (log} = 5.3761\ 5810) \\ \text{Azimuth from } A \text{ to } B &= 303°\ 40'\ 16''.608 \end{aligned}$$

Problem 7. Compute the value of the neglected term in Dalby's Theorem (Art. 122*c*) when the distance is 200 miles, $\phi = 40°$ and $\alpha = 45°$.

Problem 8. Calculate the plane coordinates of the point *L* (Problem 4, p. 205) referred to the point *N* as an origin, and employing the Tables and formulas of Art. 129. Verify the result by using the known azimuth and distance from *N* to *L*.

Problem 9. The position of point *A* is 44° 43′ 41″.254 N, 67° 24′ 25″.817 W. The position of *B* is 44° 41′ 50″.875 N, 67° 22′ 56″.062 W. It is desired to use these points from which to extend a survey. The distance between them and the azimuths are unknown. Compute s, α and α'.

CHAPTER VIII

FIGURE OF THE EARTH

132. Figure of the Earth.

The term "figure of the earth" may have various interpretations, according to the sense in which it is employed and the degree of precision with which we intend to define the earth's figure. When we say that the earth is spherical, we mean that the sphere is a rough approximation to the true figure, sufficiently close for many purposes. We adopt the sphere to represent this figure because it is a simple surface to deal with mathematically. When a closer approximation is required, we employ the spheroid, or ellipsoid of revolution. This figure is so near the truth that no closer approximation has ever been needed in practical geodetic operations, although an ellipsoid (three unequal axes) or an ovaloid (southern hemisphere the larger) may be nearer the truth. All the surfaces mentioned are *regular* mathematical surfaces, substituted for the true surface on account of their simplicity.

In defining the true figure it is necessary to distinguish between the topographical surface and that surface to which the waters of the earth tend to conform because they are free to adjust themselves perfectly to the forces acting upon them. It is this latter surface with which we are chiefly concerned in geodesy; the land surface is not referred to except in such questions as the effect of topography upon the direction and intensity of gravity. The true figure, called the *geoid*, is defined as a surface which is everywhere normal to the force of gravity, that is, an *equipotential* surface; and of all the possible surfaces of this class it is that particular one which coincides with the mean surface of the oceans of the earth. Under the continents

it is the surface to which the waters of the ocean would tend to conform if allowed to flow into very narrow and shallow canals cut through the land. It is necessary to suppose these canals narrow and shallow in order that the quantity of water removed may not modify the figure over the ocean areas.

Some idea of the relation of the spheroid, the geoid, and topographical surface may be gained by an inspection of Fig. 107. It will be seen that the geoidal surface coincides with the surface of the ocean, and that it intersects the spheroid at some distance

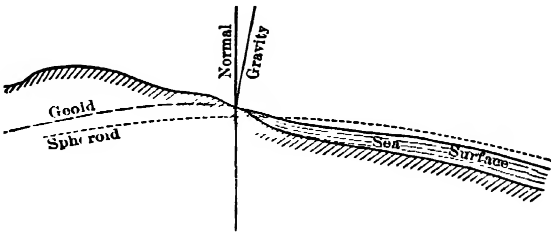

FIG. 107.

out from the shore line. The inclination of the *normal* to the *plumb line* (station error) shows the angle between the two surfaces at this point.

The surface of the geoid may be represented conveniently by means of contour lines referred to the spheroid as a datum surface. In Fig. 108, which shows contours of the geoid within the limits of the United States proper, that portion of the contours shown in full lines is taken from a map published by the Coast and Geodetic Survey in "Figure of the Earth and Isostasy" (1909); the remaining portions (dotted) were sketched in by eye, following in a general way the topography of the continent. Such a map conveys no real information about the elevations of the geoid except along the full lines, but is given simply to show how the contours would be used in representing the geoid.

When we speak of the spheroid as the "figure of the earth" we

mean that particular spheroid which best represents the earth as a whole, or which most closely fits some specified area. The dimensions of such a spheroid are not to be regarded as fixed, but are subject to revision with each accession of new data. This statement does not apply, however, to the spheroid used for the standard datum. The latter would not be changed unless it was found to be so seriously in error as to cause inconvenience in conducting the survey.

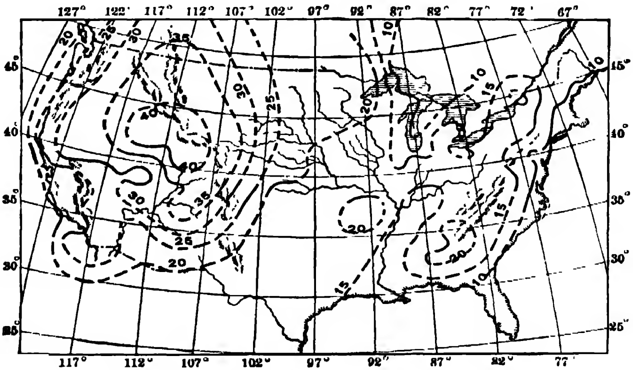

FIG. 108. Contours of the Geoid (meters).

The principal methods of determining the spheroid are (1) by the measurement of arcs, which may be portions of meridians, of parallels, or of great circles; (2) by means of areas containing several astronomical stations rigidly connected by triangulation; and (3) by observations of the force of gravity.

133. Dimensions of the Spheroid from Two Arcs.

The simplest method by which the dimensions of the spheroid can be determined is by the measurement of two meridian arcs. The length of each arc and the latitudes of the terminal points of each must be measured. If the earth were a perfect spheroid, and if there were no errors of measurement, the two arcs would determine exactly the elements of the spheroid.

In the equation of the ellipse there are two constants to be determined, and it will be shown that the determination of the curvature of the meridian ellipse at two points will enable us to compute these constants and consequently all the other elements of the ellipse. In Fig. 109, suppose that the lengths of the two meridian arcs have been measured by triangulation and that their lengths are s and s', and that the differences of the latitudes

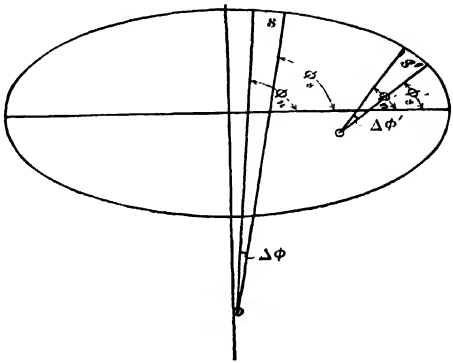

FIG. 109.

of their terminals are $\Delta\phi$ and $\Delta\phi'$, respectively. The radii of curvature of the ellipse at the middle points of the arcs are

$$R_m = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \phi)^{\frac{3}{2}}}$$

and
$$R_m' = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \phi')^{\frac{3}{2}}},$$

in which ϕ and ϕ' refer to the middle points of the arcs and a and e are unknown. If the two arcs are regarded as arcs of circles whose radii are to be found, then

$$R_m = \frac{s}{\Delta\phi \text{ arc } 1'} \quad \text{and} \quad R_m' = \frac{s'}{\Delta\phi' \text{ arc } 1'}$$

are the two radii of curvature, $\Delta\phi$ being in seconds. The shorter the arcs, the less the error involved in assuming that they are circular.

Equating the two values of R_m and R_m', we have

$$\frac{s}{\Delta\phi \text{ arc } 1''} = \frac{a\,(1 - e^2)}{(1 - e^2 \sin^2 \phi)^{\frac{3}{2}}} \tag{a}$$

and

$$\frac{s'}{\Delta\phi' \text{ arc } 1''} = \frac{a\,(1 - e^2)}{(1 - e^2 \sin^2 \phi')^{\frac{3}{2}}}. \tag{b}$$

Dividing (a) by (b) and solving for e^2,

$$e^2 = \frac{-\left(\dfrac{s\,\Delta\phi'}{s'\,\Delta\phi}\right)^{\frac{2}{3}}}{\sin^2 \phi' - \left(\dfrac{s\,\Delta\phi'}{s'\,\Delta\phi}\right)^{\frac{2}{3}} \sin^2 \phi}. \tag{81}$$

Having found e^2 from Equa. [81], the equatorial radius a may be computed by substituting the value of e^2 in either (a) or (b). The value of b may then be found from the relation

$$b^2 = a^2\,(1 - e^2). \tag{c}$$

The compression f is given by

$$f = \frac{a - b}{a}. \tag{53}$$

The length of a quadrant of the meridian may be found by applying Equa. [54], Chapter V.

In this method of determining the elements of the spheroid it should be observed that there are just enough measurements to enable us to solve the equations, and no more. All errors of measurement and of local attraction affect the result; so we should not expect to derive very accurate values from two arcs.

As an illustration of the preceding method let us take the Peruvian Arc and a portion of the Russian Arc, the data for which are as follows;

PERUVIAN ARC

Station.	Astr. lat.	Dist. in meters between the parallels of latitude.
	° ′ ″	
Tarqui.	S 3 04 32.068	344,740 5
Cotchesqui.	N 0 02 31.387	

RUSSIAN ARC (Northern End)

	° ′ ″	
Tornea	N 65 49 44.57	539,841 7
Fuglenaes. . . .	N 70 40′ 11.23	

Substituting in Formulæ [81], (*a*) and (*c*), the resulting values are

$$e^2 = 0.0065473,$$
$$a = 6,377,352\ m,$$
$$b = 6,356,440\ m.$$

134. Oblique Arcs.

If an arc (AB, Fig. 110) is inclined to the meridian at a small angle, it may be utilized to determine the curvature of the meridian as follows: Referring to Equa. (n), Chapter VII, it is seen that the difference in latitude of the terminal points of the line is given by the series for $\Delta\phi''$. Hence the length of the meridian arc is given by $\Delta\phi''.\ R_m \cdot \text{arc } 1''$, and

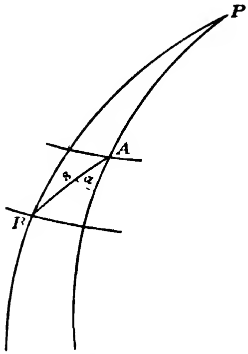

FIG. 110.

$$\Delta\phi'' \cdot R_m \cdot \text{arc } 1' = -s\cos\alpha - \frac{1}{2N}s^2\sin^2\alpha\tan\phi$$
$$+\frac{1}{6N^2}s^3\sin^2\alpha\cos\alpha(1+3\tan^2\phi). \quad [82]$$

Each line of a chain of triangles may be projected onto the meridian, and its length found by this formula. The length and difference in latitude of the end points are thus found, and the projection treated as though it were a measured meridian arc.

The sum of all these short arcs may then be treated as a single arc to be combined with another similar arc in the computation of a and e.

135. Figure of the Earth from Several Arcs.

When several arcs are to be used to determine the elements of the spheroid, there are more data than are necessary for the direct solution as given in Art. 133. The arcs usually consist of several sections; that is, the latitudes of several stations along the same meridian arc observed and the distances between them are determined by the triangulation. The problem is one of combining all these measurements by the method of least squares in order to obtain the most probable values of the elements. Only the outline of the method can be given here.

From Equa. [49] we have for the length of a meridian arc

$$s = \Delta\phi \cdot R_m \cdot \text{arc } 1'',$$

which is sufficiently accurate for short arcs. For long arcs a more accurate expression is necessary. Suppose that an arc consists of several sections, the latitude of the initial point being ϕ_1, the second ϕ_2, etc., and that the meridian distances between the stations are s, s_1, etc. From the first two latitudes

$$\phi_2 - \phi_1 = \frac{s}{R_m \text{ arc } 1''}, \tag{e}$$

in which

$$\frac{1}{R_m} = \frac{(1 - e^2 \sin^2 \phi)^{\frac{3}{2}}}{a\,(1 - e^2)}. \tag{f}$$

Instead of finding a and e^2 directly, it is more convenient to assume approximate values of these quantities and to compute the most probable corrections. Let us assume the equations

$$a = a_0 + \delta a$$

and

$$e^2 = e_0^2 + \delta e^2.$$

Let R_o be the value of R_m corresponding to e_0^2 and a_o. Ex-

panding (f) by Taylor's theorem,

$$\frac{1}{R_m} = \frac{1}{R_o} + \frac{d\left(\frac{1}{R_m}\right)}{da} \cdot \delta a + \frac{d\left(\frac{1}{R_m}\right)}{de^2} \cdot \delta e^2 + \cdots \qquad (g)$$

Evaluating the two differential coefficients,

$$\frac{d\left(\frac{1}{R_m}\right)}{da} = -\frac{(1 - e^2 \sin^2 \phi)^{\frac{3}{2}}}{a^2 (1 - e^2)} = -\frac{1}{a^2},$$

neglecting the e^2 terms, and

$$\frac{d\left(\frac{1}{R_m}\right)}{de^2} = -\frac{a(1 - e^2) \cdot \frac{3}{2} \cdot (1 - e^2 \sin^2\phi)^{\frac{1}{2}} \sin^2\phi - (1 - e^2 \sin^2\phi)^{\frac{3}{2}} \cdot a}{a^2 (1 - e^2)}$$

$$= \frac{1}{a}(1 - \tfrac{3}{2} \sin^2 \phi), \text{ neglecting } e^2 \text{ terms.}$$

Substituting these values in (g),

$$\frac{1}{R_m} = \frac{1}{R_o} - \frac{1}{a_o^2} \cdot \delta a + \frac{1}{a_o} (1 - \tfrac{3}{2} \sin^2 \phi) \cdot \delta e^2.$$

Hence (e) becomes

$$\phi_2 - \phi_1 = \frac{s}{\text{arc } 1''}\left(\frac{1}{R_o} - \frac{\delta a}{a_o^2} + (1 - \tfrac{3}{2} \sin^2 \phi) \frac{\delta e^2}{a_o}\right). \qquad (h)$$

The errors in the measured latitudes are so large in comparison with the errors in the measured arcs that the lengths are considered exact and the observed latitudes are given corrections v_1, v_2, etc. Equa. (h) then becomes

$$\phi_2 + v_2 - \phi_1 - v_1 = \frac{s}{\text{arc } 1''}\left(\frac{1}{R_o} - \frac{\delta a}{a_o^2} + (1 - \tfrac{3}{2} \sin^2 \phi) \frac{\delta e^2}{a_o}\right) \qquad (i)$$

In the small terms, containing δa and δe^2, the e^2 terms were omitted; that is, e^2 was placed equal to zero. This makes $R_m = a$ and $\phi_2 - \phi_1 = \dfrac{s}{a \text{ arc } 1''}$ in these terms.

Substituting in (i),

$$v_2 - v_1 = -(\phi_2 - \phi_1) + \frac{s}{R_o \text{ arc } 1''} - \frac{s \cdot \delta a}{a_o^2 \text{ arc } 1''} + (1 - \tfrac{3}{2}\sin^2\phi)\frac{\delta e^2}{a_o}\frac{s}{\text{arc } 1''}$$

$$= -\frac{\phi_2 - \phi_1}{a_o} \cdot \delta a + (1 - \tfrac{3}{2}\sin^2\phi)(\phi_2 - \phi_1) \cdot \delta e^2 + \frac{s}{R_o \text{ arc } 1''} - (\phi_2 - \phi_1). \quad (j)$$

If we place

$$x = \delta a,$$

$$y = \delta e^2,$$

substituting in (j), we have

$$a_1 x + b_1 y + l_1 = v_2 - v_1, \qquad (k)$$

where

$$a_1 = -\frac{\phi_2 - \phi_1}{a_o},$$

$$b_1 = (\phi_2 - \phi_1)(1 - \tfrac{3}{2}\sin^2\phi),$$

$$\frac{s_1}{R_o \text{ arc } 1''} - (\phi_2 - \phi_1).$$

It is evident that an equation of this form (k) may be written for each section of each arc. There will be more equations than there are unknown quantities to be found. From these equations we may form a set of "normal" equations (Art. 201, p. 365), equal in number to the number of unknown quantities, that is, equal to the number of arcs plus two. The simultaneous solution of the normal equations gives the corrections δa and δe^2, and also the correction to the initial latitude of each arc.

136. Principal Determinations of the Spheroid by Arcs.*

The spheroids which have been most extensively used are those of Bessel (1841) and Clarke (1866). Bessel's determination was based on the following arcs; the Peruvian, French, English, Hanoverian, Danish, Prussian, Russian, Swedish, and two Indian arcs. The resulting elements of the spheroid are generally used in Europe at the present time in geodetic surveys. They

* For an account of the different arc measurements see *A History of the Determination of the Figure of the Earth from Arc Measurements*, by A. D. Butterfield, Worcester, 1906

were employed in the United States up to about 1880. Clarke's spheroid (1866) was calculated from the following six arcs, the total amplitude being about 76° 35′; the French, English, Russian, South African, Indian, and the Peruvian. The Clarke spheroid is larger and flatter than Bessel's. It was adopted by the Coast and Geodetic Survey about 1880, after it became evident that the surface in this part of the globe has a flatter curvature than that indicated by the Bessel spheroid. The semiaxes of these two spheroids are shown below, their dimensions being based on Clarke's value of the meter, namely, $1^m = 39.370432$ inches.*

	a (meters).	*b* (meters).
Bessel (1841) .	6 377 397	6 356 079
Clarke (1866) .	6 378 206	6 356 584

Several other spheroids have been calculated from different groups of arcs, but have not been extensively used for geodetic purposes.

137. Geodetic Datum.

The question of where to place the spheroid with respect to the station points of a survey, and the question whether a certain spheroid properly represents the curvature of the area being surveyed, are determined by a comparison of the geodetic and astronomical positions of the survey points. As the survey progresses the geodetic latitudes and longitudes will be calculated on the surface of the adopted spheroid, starting from some assumed position of one of the triangulation stations. At the same time the positions of many of the stations will be determined astronomically. The differences in the latitudes, astronomical minus geodetic $(A - G)$, the differences in the longitudes, and the differences in the azimuths are computed for every

* See a report on "The Relation of the Yard to the Meter," Coast Survey Report for 1890.

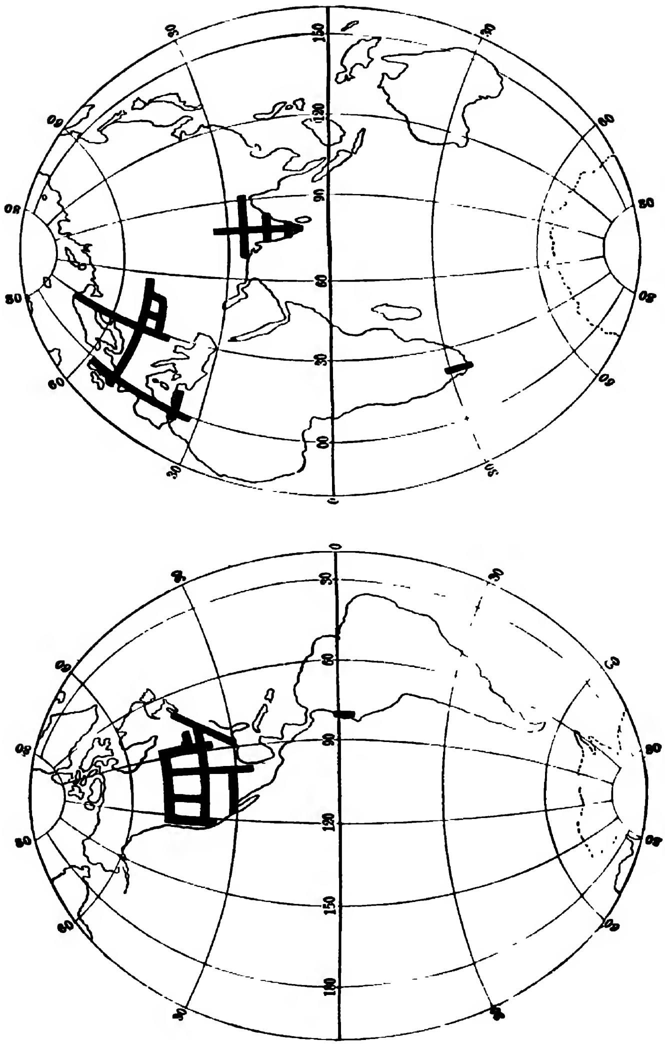

FIG. III.

station where the astronomical observations have been made. A study of these differences and their manner of distribution will show what corrections to the assumed position of the initial point will reduce the algebraic sum of the quantities $(A - G)$ to a minimum. If these differences were due wholly to errors in the assumed latitude and longitude of the initial point, it would be possible to reduce $\sum(A - G)$ to zero, but a part of this difference is due to local deflection of the vertical, that is, to the difference in slope of the geoidal and spheroidal surfaces. For this reason the most that can be expected is to place the spheroid so as to reduce $\sum(A - G)$ to a small quantity. The remaining values of $(A - G)$ at the different stations after a recomputation has been made, serve to indicate the slope of the geoid with reference to the spheroid.

If the reference spheroid adopted has too great a curvature, the computed latitudes will increase or decrease faster than the

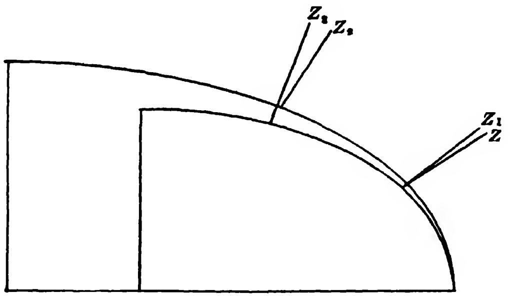

Fig. 112.

astronomical latitudes as the survey proceeds north or south from the initial point (Fig. 112). This was observed as the surveys in this country were gradually extended on the Bessel spheroid. If we consider an area instead of a meridian arc, then we see that if all the astronomical zeniths are swung inward with reference to the geodetic zeniths, the spheroid that we are using for the calculations must have too great a curvature for the area

in question. If the observed latitudes are sometimes too great, sometimes too small, as we proceed along a meridian, this simply shows that the verticals are deflected locally, and that the average curvature of the surface is nearly that of the spheroid.

138. Determination of the Geoid.

The form of the geoid is determined by observing the local variations from the spheroid as a surface of reference. These deviations may be determined either from the station error (difference between astronomical and observed position) or from the observed variation in the force of gravity.

The station error at any point, or local deflection of the vertical, is a direct measure of the slope of the surface of the geoid with reference to the spheroid. The geodetic coördinates of the point are computed with reference to a line normal to the spheroid, while the astronomical coördinates are referred to the actual direction of the plumb line, which is normal to the geoidal surface.

139. Effect of Masses of Topography on the Direction of the Plumb Line.

The deflection of the plumb line by masses of topography may be computed by applying Newton's law of gravitation, that is, if m_1 and m_2 be any two masses, D the distance between them, and k a constant (to be found by experiment), then the force of attraction between m_1 and m_2 is

$$k \cdot \frac{m_1 \cdot m_2}{D^2};$$

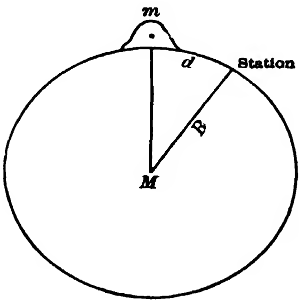

FIG. 113.

that is, the force of attraction is proportional to the product of the masses and varies inversely as the square of the distance between them. The effect of any mass, such as a mountain, in deflecting the direction of gravity at any station may be found by combining the attraction of the mountain with the

attraction of the earth regarded as a sphere. It may be shown that the attraction of a sphere at any external point is the same as though its mass were concentrated at its center. The relative attractions of the mountain and the earth upon the plumb bob at the station are as $\frac{m}{d^2}$ to $\frac{M}{R^2}$ (Fig. 113), where m is the mass of the mountain, M that of the earth, and d the distance of the mountain

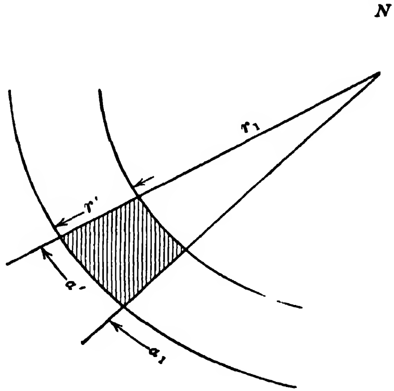

FIG. 114.

from the station. The angle D through which the plumb-line is deflected is given by

$$\tan D = \frac{mR^2}{Md^2}.$$

The earth's mass is $\frac{4}{3}\,\pi R^3 \times 5.58$ (the constant 5.58 being the mean density of the earth). If the mountain has a volume v and density δ, and the earth's radius be taken as 6370 kilometers, then

$$D' \quad 0.00138\,\frac{v\delta}{d^2}, \qquad [83]$$

the dimensions being in meters and the angle in seconds.

In order to take into account all of the topography about a station when computing the deflection of the plumb line, the following method may be employed (see Clarke, *Geodesy*, p. 294). The area surrounding the station is supposed to be divided into circular rings of any desired width, and the rings cut into four-sided compartments by radial lines, as in Fig. 114.

It is desirable to separate the component of the deflection in the meridian plane from that in the prime vertical. Let h be the height of the upper surface of the mass above station O; let α and r be the azimuth and horizontal distance to any particle P in the mass; and let z be its height above O and δ its density. The mass of the particle is then $\delta \cdot r \cdot d\alpha \cdot dr \cdot dz$. The attraction of the particle on O is

$$k \cdot \frac{\delta \cdot r \cdot d\alpha \cdot dr \cdot dz}{r^2 + z^2}$$

k being the gravitation constant.*

The component of this attraction in the plane of the meridian is the total attraction multiplied by the cosine of the angle between PO and SO, which is $\dfrac{r \cos \alpha}{\sqrt{r^2 + z^2}}$.

The total attraction of the mass in the compartment in the direction SO is

$$\begin{aligned} A &= k \int_{\alpha_1}^{\alpha'} \int_{r_1}^{r'} \int_0^h \frac{\delta \cdot r^2 \cdot \cos \alpha \, d\alpha \cdot dr \cdot dz}{(r^2 + z^2)^{\frac{3}{2}}} \\ &= k \cdot \delta \, (\sin \alpha' - \sin \alpha_1) \int_{r_1}^{r'} \int_0^h \frac{r^2 \, dr \, dz}{(r^2 + z^2)^{\frac{3}{2}}} \\ &= k \cdot \delta \cdot h \, (\sin \alpha' - \sin \alpha_1) \int_{r_1}^{r'} \frac{dr}{\sqrt{r^2}} \\ &= k \cdot \delta \cdot h \, (\sin \alpha' - \sin \alpha_1) \log_e \frac{r' + \sqrt{r'^2 + h^2}}{r_1 + \sqrt{r_1^2 + h^2}}. \end{aligned}$$

* The gravitation constant may be defined as the attraction of one unit mass on another unit mass at a unit distance away. In the C. G. S. system this is 6673×10^{-11}.

Unless h is very large, the equation may be written with sufficient accuracy

$$A = k\,\delta h\,(\sin\alpha' - \sin\alpha_1)\log_e\frac{r'}{r_1};$$

that is, the mass is considered to lie in the plane of the horizon of the station.

The attraction of the earth at point O, supposing it to be a sphere of radius R (3960 miles) and of density Δ, is

$$\begin{aligned} A' &= k\,\frac{\Delta\cdot\frac{4}{3}\,\pi R^3}{R^2} \\ &= k\cdot\tfrac{4}{3}\,\pi R\Delta. \end{aligned}$$

The angle of deflection in the plane of the meridian is given by the ratio of attractions, that is,

$$\begin{aligned} D &= \frac{\delta}{\Delta}\cdot\frac{h\,(\sin\alpha' - \sin\alpha_1)\log_e\dfrac{r'}{r_1}}{\frac{4}{3}\cdot\pi\cdot R} \\ &= 12''.44\,\frac{\delta}{\Delta}\cdot h\cdot(\sin\alpha' - \sin\alpha_1)\log_e\frac{r'}{r_1}. \end{aligned} \qquad [84]$$

The ratio of densities $\dfrac{\delta}{\Delta}$ may be taken as $\dfrac{1}{2.09}$;* $\delta = 2.67$ and $\Delta = 5.576$.

By extending the rings outward this computation may be carried as far from the station as desired. If a compartment is very far from the station, it becomes necessary to correct for the curvature of the earth, because the mass no longer lies in the horizon of the station, as at first assumed.

If the angles α_1 and α' are measured from the prime vertical instead of from the meridian, the formula gives the deflection in a plane at right angles to the meridian.

By the foregoing process we may compute for any station what is called the topographic deflection. It shows what the deflection of the plumb line would be if no other forces acted

* See Harkness, *The Solar Parallax and its Related Constants*, Washington, 1891.

upon it than those mentioned. A comparison of the values so computed with the station errors actually observed shows the former to be much larger than the latter; from which we infer that the attraction of the surface topography cannot be the only force tending to deflect the plumb line.

Laplace Points.

As stated above, it is customary to resolve the deflection of the plumb line into two components, one in the plane of the meridian and the other in the plane of the prime vertical. The meridian component is found directly by subtracting the geodetic (computed) latitude from the observed astronomic latitude. The prime vertical component must be obtained indirectly either from the astronomic and geodetic longitudes or from the astro-

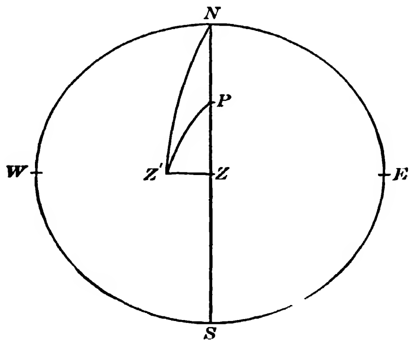

FIG. 115.

nomic and geodetic azimuths. In terms of the longitudes this component is

$$\text{p. v. component} = (\lambda_A - \lambda_G) \cos \phi_G.$$

In terms of the azimuth it is

$$\text{p. v. component} = -(\alpha_A - \alpha_G) \cot \phi_G.$$

Both of these relations may be derived from the figure (115). If we equate the two values for the prime vertical component

we obtain

$$(\alpha_A - \alpha_G) = -(\lambda_A - \lambda_G)\sin\phi_G$$

which is known as the Laplace equation. Triangulation stations at which the astronomic longitude and azimuth have been observed are called Laplace points.

The geodetic and astronomic longitudes in the United States are subject to probable errors of less than 0″.5. The astronomic azimuths are also determined with about the same accuracy. The geodetic azimuths, however, as carried through the triangulation, are subject to an error about ten times as great. The triangulation may therefore be greatly strengthened by correcting the geodetic azimuths at Laplace points by means of the above equation.

The manner of correcting the geodetic azimuth is illustrated by the following example, taken from *Supplementary Investigation in* 1909 *of the Figure of the Earth and Isostasy.*

U. S. Standard longitude of Parkersburg	=	88° 01′ 49″.00
Astronomic " " "	=	88 01 48 .30
$A - G$ in longitude		−0″ 70
$A - G$ in azimuth $= (-0.70)(-\sin\phi_g)$	=	+0 44
Astronomic azimuth Parkersburg to Denver	=	143 16 15 .55
True geodetic azimuth Parkersburg to Denver	=	143 16 15 .11
U. S Standard azimuth Parkersburg to Denver	=	143 16 15 .64
Correction to U. S. Standard azimuth	=	−0″.53

140. Isostasy — Isostatic Compensation.

For many years it has been known that the estimated and observed values of the station error are not in even approximate agreement, and it has long been suspected that the explanation would be found in the fact that the densities of the material immediately beneath the surface are unequal, regions of deficient density lying beneath mountain ranges, and regions of excessive density lying beneath low areas and under the ocean bottom. It is supposed that at some depth the excess above the surface is compensated by the defect below the surface, and vice versa. This condition is given the name *isostasy.* It appears that the

theory was first clearly stated by Major C. E. Dutton in 1889, and since that time it has been the subject of much study.

In 1909 and 1910 there were published by the Coast and Geodetic Survey the results of a very extensive investigation conducted by Professor J. F. Hayford, then Inspector of Geodetic Work and Chief of the Computing Division. The investigation was based primarily upon the computation of the topographic deflections at a large number of astronomical stations in the United States. The best topographic maps available were used for this purpose. These computed deflections were then compared with the known (observed) deflections at these same stations as found from the triangulation and astronomical observations. In substantially all cases the computed deflection was found to exceed the observed deflection by a large amount, although the two were usually of the same algebraic sign. Computations were then made to test the theory that this condition called isostasy actually exists.

The condition known as isostasy may be stated as follows: the *mass* in any prismatic column which has for its base a unit area of the horizontal surface lying at the depth of compensation, for its edges vertical lines (lines of gravity), and for its upper limit the actual irregular surface of the earth (or the sea surface if the area in question is beneath the ocean), is the same as the mass in any other similar prismatic column having a unit area on the same surface for its base. Such prismatic columns have different heights but the same mass, and their bases are at the same depth below the geoidal (sea-level) surface.

Computations were made assuming different depths of compensation, for the purpose of finding at what depth the computed deflections (taking isostasy into account) most nearly agree with the observed deflection. It was found that the compensation was most nearly complete (more than $\frac{9}{10}$ complete) at a depth of about 122 kilometers, or about 76 miles. (Later researches indicate that this should be about 60 miles).

It should be observed that, while the densities in the prismatic

columns tend to compensate, the resultant deflection of the plumb line is not zero, for the portions of the column nearest the station have a much greater influence than the distant portions. The tendency is to throw all the zeniths outward from the continental dome, assigning to the surface a curvature which is greater than it should be.

This investigation not only included a determination of the most probable depth of compensation, and a substantial proof of the validity of the theory in so far as it applies to the United States, but also included a determination of the most probable dimensions of the spheroid for that area. In this calculation the *area method* was employed. The dimensions of the spheroid resulting from this investigation are as follows:

$$a = 6{,}378{,}388^m \pm 18^m,$$
$$b = 6{,}356{,}909^m,$$
$$\frac{1}{f} = 297.0 \pm 0.5.$$

The general conclusions in regard to the existence of isostasy within the limits of the United States were later confirmed by the results of a similar investigation of the compensating effect upon observed values of the force of gravity determined with the pendulum.

The results of these investigations will be found in the following publications of the United States Coast Survey:

John F. Hayford, *The Figure of the Earth and Isostasy from Measurements in the United States*, 1909.

John F. Hayford, *Supplementary Investigations in* 1909 *of the Figure of the Earth and Isostasy*, 1910.

John F. Hayford and William Bowie, *The Effect of Topography and Isostatic Compensation upon the Intensity of Gravity*, Special Publication No. 10, 1912.

William Bowie, *The Effect of Topography and Isostatic Compensation upon the Intensity of Gravity*, Special Publication No. 12, 1912.

William Bowie, *Investigation of Gravity and Isostasy*, Special Publication No. 40, 1917.

THE AREA METHOD

The area method, already referred to, differs from the usual method, or arc method, in that no attention is given to whether the astronomic stations are located approximately on arcs, such as meridians, parallels, or obliques. Stations in any position may be used provided they are connected with each other by continuous triangulation, all computed on one basis, that is, all on the same reference spheroid and all referred to the same initial latitude, longitude and azimuth. Astronomic latitudes, longitudes, and azimuths all appear in one set of equations.

The method consists in stating for each observed astronomic latitude a "condition equation" (see Art. 203) of the form,

$$k_1(\phi) + l_1(\lambda) + m_1(\alpha) + n_1\left(\frac{a}{100}\right) + o_1(10\,000\,e^2) + (\phi_A - \phi') = D_M.$$

For each longitude observation there is an equation of the form,

$$k_2(\phi) + l_2(\lambda) + m_2(\alpha) + n_2\left(\frac{a}{100}\right) + o_2(10\,000e^2) + \cos\phi'(\lambda_A - \lambda') = D_P.$$

and for each azimuth observation, one of the form,

$$k_3(\phi) + l_3(\lambda) + m_3(\alpha) + n_3\left(\frac{a}{100}\right) + o_3(10\,000e^2) - \cot\phi'(\alpha_A - \alpha') = D_P.$$

These equations connect the observed deflections of the vertical with the constants expressing the shape and size of the earth.

In these equations the quantities ϕ_A, λ_A, α_A are the observed astronomic latitude, longitude and azimuth; ϕ', λ', α' represent the geodetic latitude, longitude, and azimuth as computed on the U. S. Standard Datum; $\phi_A - \phi'$ is the deflection in the plane

of the meridian, $\cos \phi' (\lambda_A - \lambda')$ is the prime vertical component of the deflection at a longitude station, and $\cot \phi' (\alpha_A - \alpha')$ is the prime vertical component of the deflection of the vertical at an azimuth station as found from the observed azimuth. The quantities (ϕ), (λ) and (α) are unknown corrections to the initial latitude, longitude and azimuth of Meades Ranch. The corrections to the values of a and e^2 of the Clarke Spheroid are indicated by $\left(\frac{a}{100}\right)$ and $(10\,000\, e^2)$. The coefficients of these unknowns, k_1, l_1, etc., are computed from special formulæ so as to show the effect of (ϕ), (λ), etc., at each station. For example, k_1 is a numerical coefficient such that if the initial latitude (at Meades Ranch) were corrected by the amount (ϕ) the change produced in $\phi_A - \phi'$ would be $k_1 (\phi)$. Similarly k_2 is a coefficient such that if the initial latitude were corrected by (ϕ) the change produced in $\cos \phi' (\lambda_A - \lambda')$ would be $k_2(\phi)$. The quantities in the right hand numbers of the equations are the residuals of the equations, and represent the final unexplained meridian component, and prime vertical component, respectively, of the deflections of the vertical.

The problem is to find the most probable values of the quantities (ϕ), (λ), (α), $\left(\frac{a}{100}\right)$, and $(10\,000\, e^2)$, that is the values which will make the sum of the squares of the residuals a minimum. This is effected by the "method of least squares" (see Chap. XII). A solution of the "normal equations" derived from these observation equations will give the most probable values of the five unknown quantities.

In the investigation of isostasy the depth of compensation was really a sixth unknown sought in the solution. Instead of including it as a sixth unknown in the equations, five solutions were carried out for five different assumed depths of compensation, and that depth adopted which showed the sum of the squares of the residuals to be a minimum. In these five solutions the only difference was in the values of the absolute terms.

In one solution it was assumed that there is no isostatic compensation and the absolute term should be the observed deflection of the vertical minus the computed topographic deflection. In another solution it was assumed that the compensation is complete at the surface of the earth, and that the absolute term should be the observed deflection, just as in the equations given above. The other three solutions were for assumed depths of 162.2, 120.9, and 113.7 kilometers respectively. The most probable depth was estimated to be 112.9 kilometers. Later work (in 1909) taking into account all the Laplace azimuths available, indicated that this depth should be 122 kilometers. A complete account of the investigation will be found in the reports already cited.

Since these original reports were published the matter has received much attention and further researches have been carried on. At the present time (1929) it is considered that the depth of compensation is probably 60 miles.

Another interesting result of this investigation was the contour map of the geoid referred to the Clarke Spheroid of 1866 and the U. S. Standard Datum. The contours were constructed by a purely mechanical process, quite similar to that used by military topographers. In sketching contours by means of the slope board and the plane table no actual elevations are found, but merely the slope of the surface from point to point. This gives the contour spacing. In contouring the geoid the slopes were given directly by the deflections of the vertical and the process of spacing the contours was reduced to a mechanical one which did not depend in any way upon the actual topography of the region. The results are shown by the full lines in Fig. 108.

PROBLEMS

Problem 1. Compute the dimensions of the spheroid from the following arcs.

Name.	Lat. of middle point.	Amplitude.	Length in feet.
	° ′ ″	° ′ ″	
Peruvian (Delambre's)	S 1 31 00	3 07 03.1	1 131 057
English	N 52 35 45	3 57 13.1	1 442 953

Problem 2. Compute the dimensions of the spheroid from the following arcs.

Station.	Latitude.	Distance in meters.
	° ′ ″	
Formentera	N 38 39 53 17	1 374 587
Dunkirk	N 51 02 08 41	
Tarqui	S 3 04 32 07	344 740.5
Cotchesqui	N 0 02 31 39	

Problem 3. Lake Superior arc; latitudes, 38° 43′ 17″.22 and 48° 07′ 06″.62; dist., 1,043,974 meters. Peruvian arc; latitudes, −3° 04′ 32″.0, +0° 02′ 31″.4; dist., 344,736.8 meters. Compute a and e^2.

CHAPTER IX

GRAVITY MEASUREMENTS

141. Determination of Earth's Figure by Gravity Observations. The determination of the force of gravity by means of pendulums affords a means of determining the earth's figure, which is entirely independent of the arc and area methods previously discussed. In this method the force of gravity is measured at points of known latitude and longitude. From the observed variation of gravity with the latitude the polar compression may be computed. Such measurements, therefore, will give the form but not the absolute dimensions of the spheroid.

In the following discussion the term *gravity* (g) will be taken to mean the resultant obtained by combining the force of the earth's attraction due to gravitation and the centrifugal force due to the rotation of the earth.

142. Law of the Pendulum.

The relation between l, the length of a simple pendulum, P, its period of oscillation, and g, the force of gravity is given by the formula

$$P = \pi\sqrt{\frac{l}{g}} \qquad [85]$$

or, more accurately,

$$P = \pi\sqrt{\frac{l}{g}}\left(1 + \frac{h}{8\,l}\right), \qquad [86]$$

where h is the height through which the point of oscillation falls during a half oscillation.

143. Relative and Absolute Determinations.

Determinations of gravity are of two kinds:

(1) *Absolute determinations*, in which both P and l are measured and from which g may be calculated; and (2) *relative determina-*

tions, in which P is measured at two stations and the ratio of the corresponding values of g at the two places becomes known. If the time of oscillation P of the same pendulum has been observed at two stations, then

$$g_1 = \frac{l\pi^2}{P_1^2}$$

and

$$g_2 = \frac{l\pi^2}{P_2^2};$$

whence

$$\frac{g_1}{g_2} = \frac{P_2^2}{P_1^2}. \qquad [87]$$

Absolute determinations of g are far more difficult than relative determinations, owing to the practical difficulties of measuring the length l with sufficient accuracy.

Relative determinations may be made with very great accuracy, since the time of oscillation may be measured in such a manner that the personal errors of the observer have but little effect on the results.

Most of the pendulum observations for geodetic purposes are now made by the relative method, and all values of g are made to depend upon some one reliable determination of the absolute value. The relative values of g in such a system, however, still remain more accurate than the computed absolute values.

144. Variation of Gravity with the Latitude.

The approximate law governing the variation of gravity with the latitude may be expressed thus:

$$g_\phi = g_e\left(1 + \frac{g_p - g_e}{g_e}\sin^2\phi\right), \qquad [88]$$

in which g_ϕ, g_e, and g_p are values of g at latitude ϕ, at the equator, and at the pole, respectively. By means of two such equations, one for g_ϕ observed near the equator and one for g_ϕ near the pole, the two unknowns g_e and g_p may be found.

Equation [88] may be derived in a simple manner if we may neglect variations in the attraction at different parts of the sur-

face.* Suppose the earth to be a sphere of radius r, the attraction G having the same value everywhere. Then g_ϕ, the resultant of the attraction G and the centrifugal force c, is found as follows:

At the equator the centrifugal force $= c_e = \omega^2 r$.† At the pole $c_p = 0$.

Also at the equator

$$g_e = G - c_e \tag{a}$$

and at the pole $\quad g_p = G - c_p = G;$

whence $\quad g_p - g_e = c_e.$

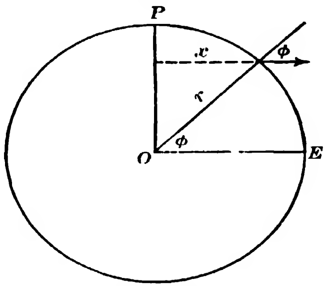

FIG 116

In latitude ϕ (Fig. 116) $x = r\cos\phi$ and $c_\phi = \omega^2 r\cos\phi = c_e\cos\phi$. The component of c_ϕ directly opposed to G is $c_e\cos^2\phi$ (vertically upward).

Hence $$g_\phi = G - c_e\cos^2\phi. \tag{89}$$

Substituting in [89] the value of G from Equa. (a),

$$\begin{aligned} g_\phi &= g_e + c_e - c_e\cos^2\phi \\ &= g_e + c_e\sin^2\phi \\ &= g_e + (g_p - g_e)\sin^2\phi; \end{aligned}$$

* See Jordan's *Handbuch der Vermessungskunde*, Vol. III, p. 627.

† The centrifugal force may be expressed by $\frac{v^2}{r}$, where v is the velocity of a particle at the equator. The distance moved by a particle in one rotation (= 1 sidereal day = T seconds) is $2\pi r$. Hence the centrifugal force $= \left(\frac{2\pi}{T}\right)^2 r = \omega^2 r$, where ω is the angular velocity. T = 86,400 sidereal seconds = 86,164.09 mean solar seconds.

that is,
$$g_\phi = g_e\left(1 + \frac{g_p - g_e}{g_e}\sin^2\phi\right). \qquad [88]$$

In order to obtain an accurate numerical expression for g_ϕ, of the same general form as the above, we may write

$$g_\phi = g_e\,(1 + B\sin^2\phi)$$

and then determine the value of B which is in best agreement with all observed values of g. For such a formula Dr. Helmert* published, in 1884, the equation

$$g_0 = 978.000\,(1 + 0.005310\sin^2\phi), \qquad [90]$$

in which g_0 is supposed to be the value at sea-level and the unit is dynes of force, or centimeters of acceleration.

This may be expressed for convenience in terms of g_0 at latitude 45°. Since $\sin^2 45° = \frac{1}{2}$,

$$g_{45} = g_e\left(1 + \frac{B}{2}\right);$$

and since
$$2\sin^2\phi = 1 - \cos 2\phi,$$
$$\sin^2\phi = \tfrac{1}{2} - \tfrac{1}{2}\cos 2\phi$$

and
$$g_0 = \frac{g_{45}}{1 + \frac{B}{2}}\left(1 + B\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\phi\right)\right)$$

$$= g_4 \qquad \frac{\frac{B}{2}}{1 + \frac{B}{2}}\cos 2\phi$$

which becomes

$$g_0 = 980.597\,(1 - 0.002648\cos 2\phi). \qquad [91]$$

In 1901 Dr. Helmert gave the more accurate forms

$$g_0 = 978.046\,(1 + 0.005302\sin^2\phi - 0.000007\sin^2 2\phi) \qquad [92]$$

and
$$g_0 = 980.632\,(1 - 0.002644\cos 2\phi + 0.000007\cos^2 2\phi), \qquad [93]$$

* Helmert, *Höhere Geodäsie*, Vol. II, p. 241.

in which the number 0.000007 ($=\frac{1}{4}B_4$) is a coefficient found theoretically from assumptions regarding the internal structure of the earth.

These formulæ refer to the absolute value of g at Vienna. To refer to the "Potsdam system," to which all values of g observed in the United States are referred,* the equations must be written

$$g_0 = 978.030\,(1 + 0.005302 \sin^2\phi - 0.000007 \sin^2 2\phi) \qquad [94]$$

and $g_0 = 980.616\,(1 - 0.002644 \cos 2\phi + 0.000007 \cos^2 2\phi)$. [95]

In the Coast Survey Special Publication No. 12, entitled "Effect of Topography and Isostatic Compensation upon the Intensity of Gravity" (second paper) the following formula is given:

$$g_0 = 978.038\,(1 + 0.005302 \sin^2\phi - 0.000007 \sin^2 2\phi), \qquad [96]$$

equivalent to

$$g_0 = 980.624\,(1 - 0.002644 \cos 2\phi + 0.000007 \cos^2 2\phi), \qquad [96a]$$

which is Helmert's formula of 1901 corrected by 0.008 dyne. The constants in these equations were derived from observations in the United States only.

In Special Publication No. 40, a study is made of observations in the United States, Canada, Europe and India. The formula resulting from this investigation is

$$g_0 = 978.039\,(1 + 0.005294 \sin^2\phi - 0.000007 \sin^2 2\phi), \qquad [97]$$

145. Clairaut's Theorem.

The relation between the flattening of the spheroid at the poles and the values of g_p and g_e is expressed by Clairaut's theorem, published in 1743, namely,

$$\frac{a-b}{a} = \frac{5}{2}\cdot\frac{c_e}{g_e} - \frac{g_p - g_e}{g_e}, \qquad [98]$$

in which c_e is the centrifugal force at the equator. In this

* The American observations for g were referred to Greenwich (England), Paris (France), and Potsdam (Germany) by observations made in 1900 by G. R. Putnam, (see Coast Survey Report for 1901).

formula the terms of the second order have been omitted. If these terms are included, the formula becomes

$$\frac{a-b}{a}=\frac{5}{2}\cdot\frac{c_e}{g_e}-B-\left(\frac{10}{3}\left(\frac{c_e}{g_e}\right)^2-\frac{17}{14}\cdot\frac{c_e}{g_e}\cdot B-\frac{B^2}{21}-\frac{2}{21}B_4\right). \quad [98a]$$

in which B and B_4 are coefficients to be determined from the observations (Helmert, *Höhere Geodasie*, Vol. II, p. 83). It is by means of this equation that the form of the earth is computed from gravity observations.

146. Pendulum Apparatus.

Nearly all of the observations of gravity for geodetic purposes are made with pendulums of invariable length, by the *relative* method. The description of apparatus in the following articles will be limited to one type, the half-seconds *invariable* pendulum apparatus as designed and constructed by the United States Coast Survey. The first half-seconds invariable pendulum with electrical apparatus for determining the period appears to have been devised by Sterneck (Austria) in 1882. In 1890 T. C. Mendenhall, then Superintendent of the Coast and Geodetic Survey, designed an apparatus of this kind but differing in many details, however, from any previous design. This apparatus has been used ever since that time in substantially the same form excepting the addition of the interferometer for determining the flexure. This apparatus includes three half-second pendulums, each about 248^{mm} long and having an agate plane at the point of suspension. The agate plane rests on a knife-edge support (angle of 130°) attached to the pendulum case in which the pendulums are enclosed when they are swung. The use of the blunt angle on the knife edge and the placing of the plane (rather than the knife edge) on the pendulum are designed to secure greater permanence of length, upon which the accuracy of the method depends. The pendulums are made of an alloy of copper and aluminum and weigh 1200 grams each. The three are of slightly different lengths so that they will have different periods. Their form (Fig. 118) is such as to give strength and at the same time

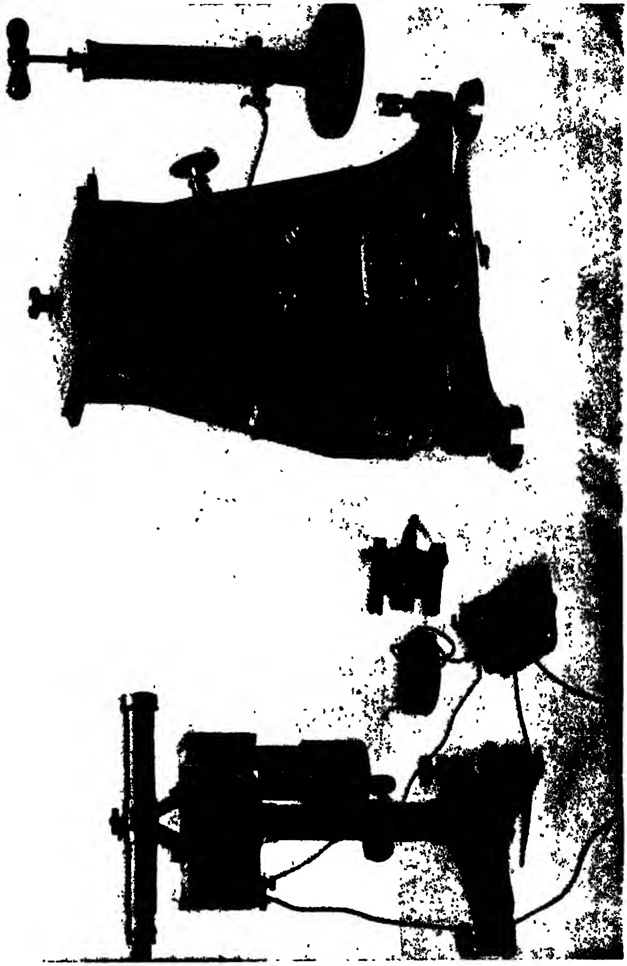

FIG.

offer but little resistance to the air. In addition to the three observing pendulums there is a dummy pendulum, of the same size and shape but carrying a thermometer packed in filings of

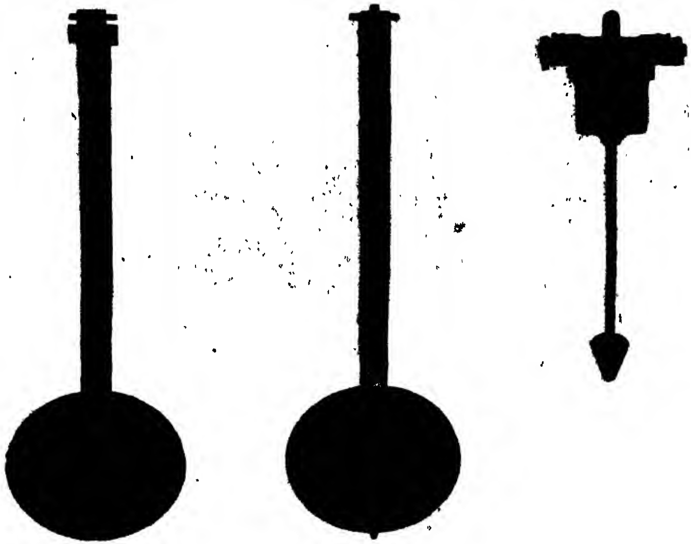

FIG. 118.

the same metal. There is also a small pendulum provided with a spirit level for leveling the knife edge.

Pendulums made of invar metal have been constructed by the instrument division of the Coast and Geodetic Survey so that it is possible to make gravity observations on mountain peaks and other places where the control of temperature is difficult. The use of this metal makes it unnecessary to construct a "constant temperature room."

The pendulums are swung in an air-tight case from which the

air may be nearly exhausted by means of a pump. Levers are provided for lowering the pendulum onto the knife edge and for starting and stopping the pendulum. Inside the case is a manometer tube for registering the air pressure, and also an additional thermometer. Levels are provided for leveling the case, and there is a graduated scale under the pendulum for reading the arc of oscillation. In the most recent work of the Coast Survey the pendulum receiver has been enclosed in a felt and leather case to prevent fluctuations in temperature.

The observations are made by comparing the times of oscillation of the pendulums with the half-second beats of a break-circuit (sidereal) chronometer connected electrically with the "flash apparatus" used for observing the coincidence.

The flash apparatus (Fig. 119) consists of a shutter *a* operated by the armature of an electromagnet *b* in the circuit and a mirror

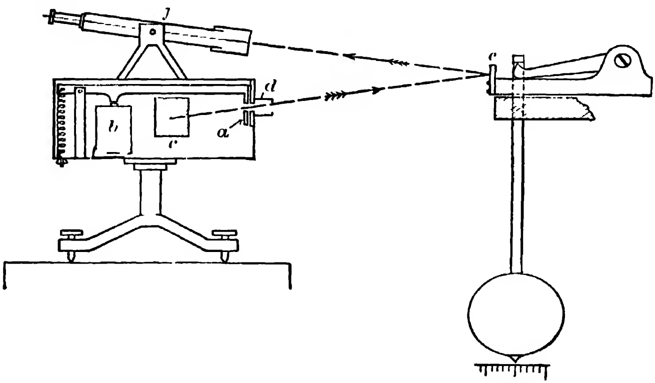

FIG. 119.

c behind the shutter which reflects light through the slit *d* to two small mirrors *c*, which reflect it into an observing telescope *f*; one of the small mirrors is attached to the knife-edge support. In the most recent form of the flash apparatus, the observer looks down through a vertical tel-

escope and sees the flash reflected by a prism. This arrangement is more convenient for the observer than the older form because the pendulum receiver is usually mounted on a very low support.

When the pendulum is at rest and the shutter open, a beam of light from a lamp* at one side of the apparatus strikes the mirror c at an angle of 45° and passes through the slit; it is reflected from both mirrors at c and appears to the observer as two horizontal bright slits side by side. The mirrors may be adjusted so that these slits appear to be at the same height, so as to form one continuous band. If the pendulum is set swinging, the reflected image now appears to travel up and down, while the image from the other mirror is stationary. If the shutter is closed and allowed to open only for an instant at the end of each second (or each two seconds), the observer sees that at each successive opening of the shutter the moving image has changed its position relative to the fixed image. This is due to the fact that the period of the pendulum is longer than the sidereal second and the pendulum has made slightly less than one complete (double) oscillation. By watching the flashes and noting the chronometer readings when they coincide, the observer obtains the number of seconds between two successive coincidences. During this interval the pendulum has evidently lost just one oscillation on the (half-second) beats of the chronometer. In the interval between two successive coincidences the pendulum has made one less than twice as many oscillations as the chronometer has beat seconds. During the interval between any two coincidences the number of oscillations is twice the number of seconds (s) less the number of coincidence intervals (n). Hence the time of one oscillation (P) is given by

$$P = \frac{s}{2s - n}. \qquad [99]$$

* An electric bulb placed inside the flash box is now used instead of the oil lamp.

An examination of this formula will show that an error in noting the times of coincidence produces a relatively small error in P, and for this reason the method is almost independent of the observer's errors.

On account of the variation of g (and consequently of P) with the latitude of the station, it is necessary to use a mean-time chronometer at stations situated near the pole, because the period of the pendulum approaches so closely to the sidereal half-second that the coincidence intervals are inconveniently long. In case a mean-time chronometer is used, the formula becomes

$$2s + n \qquad [100]$$

147. Apparatus for Determining Flexure of Support. Observations with pendulums mounted on a very flexible support show plainly that when a pendulum is set swinging, it communicates motion to the case and the support and sets them oscillating, and this oscillation in turn affects the observed period of the pendulum. The apparatus now used to measure the effect of this flexure is one which operates on the principle of the interferometer.* This is an optical device (Fig. 120) consisting of a lamp and lens arranged so as to furnish a beam of sodium light; a glass plate arranged so as to separate the beam of light into two parts, one of which is transmitted, the other reflected; two mirrors, one in the path of each beam of light; and a telescope for observing the image. When the different parts of the apparatus are properly adjusted, dark and light bands will appear in the field telescope, owing to interference of the sodium-light waves of the two beams. One of the mirrors is mounted on the pendulum receiver, while the rest of the apparatus is on an independent support in front of it. When the pendulum is set swinging, it sets the case in motion, and this in turn moves the mirror, causing a slight variation in the length of the path of one

* A description of the interferometer will be found in the Coast Survey Report for 1910.

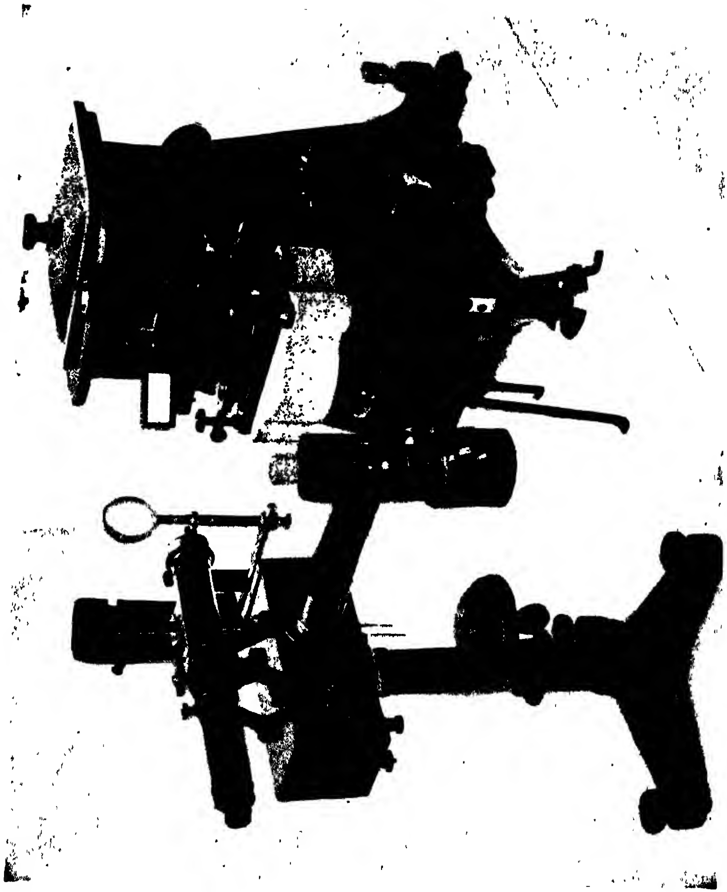

FIG. 120. Interferometer (showing relative position). During flexure observations the interferometer is on an independent support. (*C. L. Berger and Sons.*)

of the beams of light. This causes the interference bands to shift back and forth; the amount of shift may be estimated by observing the motion of the bands over a cross-hair or a scale in the field of the telescope. It is usually observed by noting the scale readings of both edges of some band in each of its two positions (before and after shifting). The movement of the edges of a band divided by the width of the band (in scale divisions) gives the movement in units of the width of a band. Figure

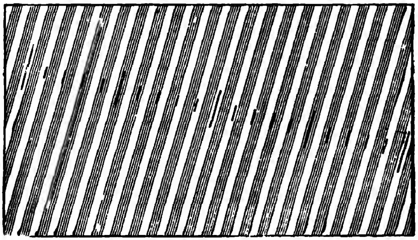

FIG. 121.

121 represents the interference (dark) bands and the scale divisions in the field of the telescope.

Tests made with the pendulum mounted on supports of different degrees of flexibility will show the relation between the observed movement of the fringe bands and the resulting error in the period of the pendulum. In the Coast Survey tests the results showed that a movement equal to the width of one band produced a change of 173 in P in units of the seventh decimal place. This is more conveniently expressed as follows: 0.01 F produces a change of 1.73 in P, where F is the width of a band. This constant was determined with the pendulum swinging through an arc of 5^{mm} on the scale, and all observed flexures must be reduced to this arc before correcting P.

148. Methods of Observing.

The receiver should be mounted on a solid support such as a cement or brick pier, the foot screws cemented to the pier, and the instrument sheltered as in case of astronomical observations.

It is important that the instrument should be so sheltered that the temperature will not fluctuate rapidly. The apparatus should be leveled by means of the spirit level on the outside of the case and then the knife edge should be leveled by means of the leveling pendulum. In moving the pendulums great care should be used to protect them from injury and to prevent any foreign matter from adhering to them. The accuracy of the results will depend upon the permanency of length, and any injury due to fall, or change of period due to change in the mass, will affect the period and vitiate the results. The pendulums should not be touched with the hands, but should be lifted by means of a special hook made for this purpose. The flash apparatus, chronometer, and interferometer should be placed upon supports that are entirely independent of the pendulum support.

Various programs of observing have been tried, but the following has been chiefly used by observers of the Coast Survey. Each of the three pendulums is swung first in the direct and then in the reversed position, making six swings each of eight hours' duration. The error of the chronometer is obtained by star-transit observations (Arts. 52–71) made just before the beginning and at the end of the series. The following table will indicate more clearly the order of operations.

Star Observations	9 10 P.M.
Start Pendulum No 1	10 P.M.
Reverse No. 1	6 A M
Start No. 2	2 P M.
Reverse No. 2	10 P M.
Start No. 3	6 A.M.
Reverse No 3	2 P M.
Star Observations	9 P M.
Stop Pendulum No. 3	after star observations

If star observations are lost at the end of the set, the swings are continued until star observations are obtained. At the beginning and end of each swing several coincidences are observed. At the end of each swing several more are observed. Very little time is lost between swings, so that they are almost continuous

between star observations. For this reason the variations in the rate of the chronometer are almost entirely eliminated from the mean result of all the swings.

Since 1913 the Coast Survey observers have obtained the chronometer corrections from the Naval Observatory time signals instead of by direct observations. This results in a great saving of time and cost. Another change in the regular program, recently introduced, is to swing the pendulums for twelve hours instead of eight, and in the direct position only, instead of direct and reversed.

After a pendulum is placed in position on its support, the case closed, and the air exhausted until the pressure is about 60^{mm}, the observer lowers the pendulum until it rests upon the knife edge, starts it swinging through an arc of about $0° 53'$, and notes the arc on the scale. To observe coincidences, the observer switches in the chronometer and the flash apparatus and then watches the flashes to see when they are approaching coincidence. As the two approach he notes the hours, minutes, and seconds on the chronometer when the advancing edge of the moving flash touches the first edge of the fixed flash. A few seconds later he notes when the receding edge of the moving flash touches the second edge of the fixed flash. The mean of the two gives the true time of coincidence of centers more accurately than it could be observed directly. Such observations are made on several successive coincidences, the flash moving alternately upward and downward. By combining the *up* and the *down* observations, errors of adjustment are eliminated. After a few of these have been recorded, the observer cuts out the chronometer and leaves the pendulum swinging for a period of nearly eight hours. Immediately after the observations for coincidences are completed, the temperatures are read on the two thermometers, and the pressure is read on the manometer tube. At the end of the eight-hour period the observer again observes a few coincidences as well as the arc (now diminished to about $0° 20'$), the pressure, and the temperatures. It is not necessary that he continue

observing throughout the whole eight-hour period, because the few observations already referred to make it possible to estimate correctly the number of coincidences which must have occurred between the observed times. It is customary to take the observations with two or more chronometers as a check.

This description applies to the 8-hour program outlined above. If the pendulums are swung for a 12-hour period it is necessary to start each pendulum with a somewhat larger arc ($1^\circ\ 27'$) in order that it may have a sufficient amplitude at the end of 12 hours to enable the observer to read the coincidences of the flash conveniently and accurately.

It is desirable that the temperature of the apparatus be kept as nearly uniform as possible, and that there be little vibration. In order to allow the pendulum time to assume the temperature of the receiver the next pendulum to be swung is placed inside the case before it is used in the observations. If invar pendulums are being used, however, this is not desirable, on account of possible magnetic effects. While the case is still in position the observer must place the interferometer in position and observe the movement of the interference bands while the pendulum is swinging.

149. Calculation of Period.

After the observations are complete and the time observations and the chronometer rates are computed, the time of one oscillation for each pendulum in each position is found as follows: divide the total number of seconds in an 8^h interval by the number of seconds found for one coincidence interval (see example), to obtain the number of intervals that have occurred during the swing. Since this must be a whole number, there will be no difficulty in determining it correctly. Then reverse the process, dividing the total interval by the number of coincidence intervals, to obtain the accurate value of the number of seconds (s) in one coincidence interval. The uncorrected period of the pendulum is found by

$$P = \frac{s}{2s - 1} \qquad [101]$$

for a sidereal chronometer, Table G, or

$$P = \frac{s}{2s + 1} \qquad [102]$$

for a mean-time chronometer.

150. Corrections.

This period must then be corrected to reduce it to its value at assumed standard conditions, namely,

Infinitesimal arc,
Temperature 15° C.,
Pressure 60^{mm} at 0° C.,
True sidereal time, and
Inflexible support.

The correction to reduce P to its value for an infinitesimal arc is

$$\frac{PM}{32} \cdot \frac{\sin(\phi + \phi')\sin(\phi - \phi')}{\log \sin \phi - \log \sin \phi'} \qquad [103]$$

a formula given by Borda, in which P = the period, M = the modulus of the common system of logarithms, and ϕ and ϕ' = the initial and final semiarcs.

A modified form of this formula is

$$-0.1924 \frac{a_0^2 - a_n^2}{\log_{10} a_0 - \log_{10} a_n}$$

in which $a_0 = 2R\phi$, $a_n = 2R\phi'$ and R is the distance from knife edge to arc scale, $= 296.93^{mm}$ for the Coast Survey pendulums. This gives the correction in units of the 7th decimal place; it is based on the assumption that $P = 0.5$.

The temperature correction is

$$\alpha P(15^\circ - T^\circ), \qquad [104]$$

T° being the observed temperature centigrade and α the coefficient to be found by trial. Substituting $P = 0.5$, the formula is $\beta(15^\circ - T^\circ)$. For the bronze pendulums β varies

from 0.000 00413 to 0.000 00419. For invar pendulums it is 0.000 00028.

The pressure correction is

$$K\left[60^{mm} - \frac{Pr}{1 + 0.00367\,T^\circ}\right], \tag{105}$$

in which Pr = observed pressure in mm,

T° = temperature centigrade,

and K = coefficient to be found by trial.

The value of K for the bronze pendulums is 0.000 000 101; for the invar it is 0.000 000 089.

The constant 0.00367 is the coefficient of expansion of air for 1° C.

The rate correction is given by the expression

$$+\ 0.000011574\,RP, \tag{106}$$

where R = daily rate of chronometer on sidereal time, + when losing and − when gaining. The coefficient is the reciprocal of the number of seconds in one day.

The flexure correction is computed by dividing the observed movement of the fringe band (in scale divisions) by the width of a band and then reducing this to an arc of 5^{mm} by dividing by the observed arc and multiplying by 5. The result is the displacement for a 5^{mm} arc in terms of the width of a band. This displacement, multiplied by the coefficient (173 mentioned before), gives the correction to be subtracted from P.

TABLE D. — REDUCTION OF SCALE READING IN MILLIMETERS TO MINUTES OF ARC

Scale	1.0 mm	2.0 mm.	3.0 mm.	4.0 mm.	5.0 mm.
mm	′	′	′	′	′
0.0	12	23	35	46	58
0.1	13	24	36	48	59
0.2	14	26	37	49	60
0.3	15	27	38	50	61
0 4	16	28	39	51	63
0 5	17	29	41	52	64
0 6	19	30	42	53	65
0 7	20	31	43	55	66
0 8	21	32	44	56	67
0 9	22	34	45	57	68

TABLE E. ARC CORRECTIONS (ALWAYS SUBTRACTIVE) FOR HALF-SECOND PENDULUMS

Arc at Beginning

Arc at end.	90′.	85′.	80′.	75′.	70′.	65′.	60′	55′.	50′.	45′	40′	35′.	30′.	25′	20′.
′															
5															
10	12 0	11 0	10 0	9 0	8 1	7 3	6 5	5 8	5 0	4 3	3 6	3 0	2 4	1 9	1 4
15	14 4	13 3	12 2	11 1	10 0	9 0	8 0	7 2	6 3	5 4	4 6	3 9	3 2		
20	16 9	15 6	14 3	13 0	11 8	10 7	9 6	8 6	7 6	6 6	5 7	4 9	4 1		
25	19 3	17 8	16 4	15 0	13 7	12 4	11 2	10 1	9 0	8 0	6 9				
30	21 7	20 1	18 5	17 0	15 6	14 2	12 9	11 6	10 4	9 2	8 1				
35	24 1	22 4	20 7	19 2	17 6	16 1	14 6	13 2	11 8						
40	26 5	24 7	22 9	21 2	19 5	17 9	16 3	14 8	13 3						
45	29 0	27 1	25 2	23 4	21 6	19 9	18 2								
50	31 5	29 4	27 4	25 5	23 6	21 8	20 0								
55	34 1	32 0	29 8	27 8	25 8										
60	36 7	34 4	32 2	30 0	27 9										
65	39 4	37 0	34 6												
70	42 1	39 6	37 1												
75	44 9														
80	47 7														
85															
90															

In practice it is convenient to combine Tables D and E into a single table computed for such intervals that little interpolation is necessary.

TABLE F. — CORRECTION FOR PRESSURE

Temp. C.	50 mm.	55 mm.	60 mm.	65 mm	70 mm	75 mm	80 mm.	85 mm	90 mm
0	+10	+5	0	−5	−10	−15	−20	−25	−30
1	+10	+5	0	−5	−10	−15	−20	−25	−30
2	+10	+5	0	−5	− 9	−14	−19	−24	29
3	11	6	+1	4	9	14	19	24	29
4	11	6	+1	4	9	14	19	24	29
5	11	6	+1	4	9	14	19	24	28
6	11	6	+1	4	9	14	19	24	28
7	11	6	2	3	8	13	18	23	28
8	11	6	2	3	8	13	18	23	27
9	12	7	2	3	8	13	17	22	27
10	12	7	2	3	8	13	17	22	27
11	12	7	2	3	7	12	17	21	26
12	12	7	2	2	7	12	17	21	26
13	12	7	3	2	7	12	17	21	26
14	12	8	3	2	7	11	16	21	26
15	13	8	3	2	6	11	16	20	26
16	13	8	3	2	6	11	16	20	25
17	13	8	4	1	6	11	15	20	25
18	13	8	4	1	6	10	15	20	24
19	13	9	4	−1	5	10	15	20	24
20	13	9	4	−1	5	10	15	20	24
21	14	9	4	−1	5	10	14	19	24
22	14	9	4	−1	5	10	14	19	23
23	14	9	5	0	5	9	14	19	23
24	14	9	5	0	4	9	14	18	23
25	14	10	5	0	4	9	13	18	22
26	14	10	5	+1	4	9	13	18	22
27	14	10	5	+1	4	8	13	17	22
28	+15	+10	+6	+1	− 4	− 8	−13	−17	22
29	+15	+10	+6	+1	− 3	−18	−12	−17	−21
30	+15	+10	+6	+1	− 3	− 8	−12	−17	−21

Body of table gives corrections (in 7th decimal place of seconds) to period of half seconds pendulum.

TABLE G. — PERIODS OF QUARTER METER PENDULUM

NOTE: To obtain period to 7th decimal place, prefix .50 or .500 to figures in the table.
Body of table gives

0	2200	2300	2400	2500	2600	2700	2800	2900	3000	3100
0	11,390	10,893	10,438	10,020	9634	9276	8944	8636	8347	8078
1	84	89	34	16	30	73	41	33	44	75
2	79	84	30	12	26	70	38	30	42	72
3	74	79	25	08	23	66	35	27	39	70
4	69	74	21	04	19	63	32	24	36	67
5	11,364	10,870	10,417	10,000	9615	9259	8929	8621	8333	8064
6	58	65	12	9996	12	56	25	18	30	62
7	53	60	08	92	08	52	22	15	28	59
8	48	55	04	88	04	49	19	12	25	57
9	43	51	10,399	84	01	46	16	09	22	54
10	11,338	10,846	10,395	9980	9597	9242	8913	8606	8320	8052
11	33	41	91	76	93	39	10	03	17	49
12	28	37	86	72	90	35	06	00	14	46
13	22	32	82	68	86	32	03	8597	11	44
14	17	27	78	64	82	28	00	94	08	41
15	11,312	10,822	10,373	9960	9578	9225	8897	8591	8306	8039
16	07	18	69	56	75	22	94	88	03	36
17	02	13	65	52	71	18	91	85	00	33
18	11,297	08	61	48	68	15	87	82	8297	31
19	92	04	56	44	64	12	84	79	95	28
20	11,287	10,799	10,352	9940	9560	9208	8881	8576	8292	8026
21	82	94	48	36	57	05	78	73	89	23
22	76	90	43	32	53	01	75	70	86	20
23	72	85	39	28	49	9198	72	68	84	18
24	66	80	35	25	46	95	68	64	81	15
25	11,261	10,776	10,331	9921	9542	9191	8865	8562	8278	8013
26	56	71	26	17	38	88	62	59	75	10
27	51	67	22	13	35	84	59	56	73	08
28	46	62	18	09	31	81	56	53	70	05
29	41	57	14	05	27	78	53	50	67	03
30	11,236	10,753	10,309	9901	9524	9174	8850	8547	8264	8000
31	31	48	05	9897	20	71	46	44	62	7997
32	26	44	01	93	17	68	43	41	59	95
33	21	39	10,297	89	13	64	40	38	56	92
34	16	34	92	85	09	61	37	35	53	90
35	11,211	10,730	10,288	9881	9506	9158	8834	8532	8251	7987

WHEN PENDULUM IS SLOWER THAN CHRONOMETER

Top and left-hand arguments combined give interval s = ten coincidence intervals
t = period in seconds.

3200	3300	3400	3500	3600	3700	3800	3900	4000	4100	4200	0
7825	7587	7364	7153	6954	6766	6588	6418	6258	6105	5960	0
22	85	62	51	52	64	86	17	56	04	58	1
20	83	59	49	50	62	84	15	55	02	57	2
17	80	57	47	48	60	82	14	53	01	55	3
15	78	55	45	46	59	81	12	52	6099	54	4
7812	7576	7353	7143	6944	6757	6579	6410	6250	6098	5952	5
10	74	51	41	42	55	77	09	48	96	51	6
08	71	49	39	41	53	76	07	47	95	50	7
05	69	46	37	39	51	74	05	45	93	48	8
03	67	44	35	37	49	72	04	44	92	47	9
7800	7564	7342	7133	6935	6748	6570	6402	6242	6090	5945	10
7798	62	40	31	33	46	69	00	41	89	44	11
96	60	38	29	31	44	67	6399	39	87	42	12
93	58	36	27	29	42	65	97	38	86	41	13
91	55	34	25	27	40	63	96	36	84	40	14
7788	7553	7331	7123	6925	6738	6562	6394	6234	6083	5938	15
86	51	29	21	23	37	60	92	33	81	37	16
83	48	27	18	21	35	58	91	31	80	35	17
81	46	25	16	19	33	56	89	30	78	34	18
78	44	23	14	18	31	55	87	28	77	33	19
7776	7542	7321	7112	6916	6730	6553	6386	6227	6075	5931	20
74	39	19	10	14	28	51	84	25	74	30	21
71	37	16	08	12	26	50	82	24	72	28	22
69	35	14	06	10	24	48	81	22	71	27	23
66	32	12	04	08	22	46	79	20	70	26	24
7764	7530	7310	7102	6906	6720	6544	6378	6219	6068	5924	25
62	28	08	00	04	19	43	76	17	66	23	26
59	26	06	7098	02	17	41	74	16	65	21	27
57	23	04	96	00	15	39	73	14	64	20	28
7754	7521	01	94	6898	13	38	71	13	62	19	29
7752	7519	7299	7092	6897	6711	6536	6369	6211	6061	5917	30
50	16	97	90	95	10	34	68	10	59	16	31
47	14	95	88	93	08	32	66	08	58	14	32
45	12	93	86	91	06	31	64	07	56	13	33
42	10	91	84	89	04	29	63	05	55	12	34
7740	7508	7289	7082	6887	6702	6527	6361	6204	6053	5910	35

TABLE G (*Con.*).—PERIODS OF QUARTER METER PENDU-

NOTE: To obtain period to 7th decimal place, prefix .50 or .500 to figures in the table.
Body of table gives

0	2200	2300	2400	2500	2600	2700	2800	2900	3000	3100
36	06	25	84	78	02	54	31	30	48	85
37	01	20	80	74	9498	51	28	27	45	82
38	11,196	16	75	70	95	48	24	24	43	80
39	91	11	71	66	91	44	21	21	40	77
40	11,186	10,707	10,267	9862	9488	9141	8818	8518	8237	7974
41	81	02	63	58	84	38	15	15	34	72
42	76	10,698	58	54	81	34	12	12	32	69
43	71	93	54	50	77	31	09	09	29	67
44	66	88	50	46	73	27	06	06	26	64
45	11,161	10,684	10,246	9842	9470	9124	8803	8503	8224	7962
46	56	79	42	39	66	21	00	00	21	59
47	51	75	38	35	62	18	8797	8498	18	57
48	46	70	33	31	59	14	94	95	16	54
49	41	66	29	27	55	11	90	92	13	52
50	11,136	10,661	10,225	9823	9452	9108	8787	8489	8210	7949
51	31	56	21	19	48	04	84	86	08	47
52	26	52	17	16	45	01	81	83	05	44
53	21	47	12	12	41	9098	78	80	02	42
54	16	43	08	08	38	94	75	78	8199	39
55	11,111	10,638	10,204	9804	9434	9091	8772	8475	8197	7936
56	06	34	10,200	9800	30	88	69	72	94	34
57	01	29	10,196	9796	27	84	66	69	91	32
58	11,096	25	92	92	23	81	63	66	89	29
59	91	20	88	88	20	78	60	63	86	26
60	11,086	10,616	10,183	9785	9416	9074	8757	8460	8183	7924
61	82	11	79	81	13	71	54	57	81	21
62	77	07	75	77	09	68	51	54	78	19
63	72	02	71	73	06	65	47	52	75	16
64	67	10,598	67	69	02	61	44	49	73	14
65	11,062	10,593	10,163	9766	9398	9058	8741	8446	8170	7911
66	57	89	59	62	95	55	38	43	67	09
67	52	84	54	58	92	51	35	40	65	06
68	47	80	50	54	88	48	32	37	62	04
69	42	75	46	50	84	45	29	34	59	01
70	11,038	10,571	10,142	9747	9381	9042	8726	8432	8157	7899

LUM WHEN PENDULUM IS SLOWER THAN CHRONOMETER

Top and left-hand arguments combined give interval s = ten coincidence intervals.
t = period in seconds.

3200	3300	3400	3500	3600	3700	3800	3900	4000	4100	4200	0
38	05	86	80	85	01	26	60	02	52	09	36
35	03	84	78	83	6699	24	58	00	50	07	37
33	01	82	76	81	97	22	56	6199	49	06	38
30	7498	80	74	80	95	21	55	97	47	05	39
7728	7496	7278	7072	6878	6693	6519	6353	6196	6046	5903	40
26	94	76	70	76	92	17	52	94	44	02	41
23	92	74	68	74	90	16	50	93	43	00	42
21	90	72	66	72	88	14	48	91	42	5899	43
18	87	70	64	70	86	12	47	90	40	98	44
7716	7485	7267	7062	6868	6684	6510	6345	6188	6039	5896	45
14	83	65	60	66	83	09	44	87	37	95	46
11	80	63	58	64	81	07	42	85	36	93	47
09	78	61	56	62	79	05	40	84	34	92	48
06	76	59	54	61	77	04	39	82	33	91	49
7704	7474	7257	7052	6859	6676	6502	6337	6180	6031	5889	50
02	72	55	50	57	74	00	36	79	30	88	51
7699	69	53	48	55	72	6499	34	77	28	86	52
97	67	51	46	53	70	97	32	76	27	85	53
95	65	48	44	51	69	95	31	74	26	84	54
7692	7463	7246	7042	6849	6667	6494	6329	6173	6024	5882	55
90	60	44	40	47	65	92	28	71	23	81	56
88	58	42	38	46	63	90	26	70	21	80	57
85	56	40	36	44	61	88	24	68	20	78	58
83	54	38	34	42	60	87	23	67	18	77	59
7680	7452	7236	7032	6840	6658	6485	6321	6165	6017	5875	60
78	49	34	30	38	56	83	20	64	15	74	61
76	47	32	28	36	54	82	18	62	14	73	62
73	45	30	26	34	52	80	16	61	12	71	63
71	43	28	24	32	51	78	15	59	11	70	64
7669	7440	7225	7022	6831	6649	6477	6313	6158	6010	5868	65
66	38	23	20	29	47	75	12	56	08	67	66
64	36	21	18	27	45	73	10	55	07	66	67
62	34	19	17	25	44	72	08	53	05	64	68
59	32	17	15	23	42	70	07	52	04	63	69
7657	7429	7215	7013	6821	6640	6468	6305	6150	6002	5862	70

TABLE G (*Con.*). — PERIODS OF QUARTER METER PENDU-

NOTE: To obtain period to 7th decimal place, prefix .50 or .500 to figures in the table.
Body of table gives

0	2200	2300	2400	2500	2600	2700	2800	2900	3000	3100
71	33	66	38	43	77	38	23	29	54	96
72	28	62	34	39	74	35	20	26	51	94
73	23	57	30	35	70	32	17	23	49	91
74	18	53	26	31	67	29	14	20	46	89
75	11,013	10,548	10,122	9728	9363	9025	8711	8418	8143	7886
76	08	44	17	24	60	22	08	15	41	84
77	04	40	13	20	56	19	05	12	38	81
78	10,999	35	09	16	53	16	02	09	35	79
79	94	31	05	12	49	12	8699	06	33	76
80	10,989	10,526	10,101	9709	9346	9009	8696	8403	8130	7874
81	84	22	10,097	05	42	06	93	01	28	72
82	79	18	93	01	39	02	90	8398	25	69
83	74	13	89	9697	35	8999	87	95	22	67
84	70	09	85	94	32	96	84	92	20	64
85	10,965	10,504	10,081	9690	9328	8993	8681	8389	8117	7862
86	60	10,500	77	86	25	90	78	86	14	59
87	55	10,495	73	82	21	86	75	84	12	57
88	51	91	68	79	18	83	72	81	09	54
89	46	87	64	75	14	80	69	78	06	52
90	10,941	10,482	10,060	9671	9311	8977	8665	8375	8104	7849
91	36	78	56	68	08	74	62	72	01	47
92	31	73	52	64	04	70	60	70	8098	44
93	27	69	48	60	01	67	56	67	96	42
94	22	65	44	56	9297	64	54	64	93	40
95	10,917	10,460	10,040	9653	9294	8961	8650	8361	8091	7837
96	12	56	36	49	90	57	48	58	88	34
97	08	52	32	45	87	54	44	56	85	32
98	03	47	28	41	83	51	42	53	83	30
99	10,898	43	24	38	80	48	39	50	80	27
100	10,893	10,438	10,020	9634	9276	8944	8636	8347	8078	7825

LUM WHEN PENDULUM IS SLOWER THAN CHRONOMETER

Top and left-hand arguments combined give interval s = ten coincidence intervals.
t = period in seconds.

3200	3300	3400	3500	3600	3700	3800	3900	4000	4100	4200	0
55	27	13	11	19	38	67	04	49	01	60	71
52	25	11	09	18	37	65	02	47	00	59	72
50	23	09	07	16	35	63	00	46	5998	58	73
48	21	07	05	14	33	61	6299	44	97	56	74
7645	7418	7205	7003	6812	6631	6460	6297	6142	5995	5855	75
43	16	02	01	10	30	58	96	41	94	53	76
41	14	00	6999	08	28	57	94	40	92	52	77
38	12	7198	97	06	26	55	92	38	91	51	78
36	10	96	95	05	24	53	91	36	89	49	79
7634	7407	7194	6993	6803	6622	6452	6289	6135	5988	5848	80
31	05	92	91	01	21	50	88	34	87	47	81
29	03	90	89	6799	19	48	86	32	85	45	82
27	01	88	87	97	17	47	85	30	84	44	83
24	7399	86	85	95	16	45	83	29	82	42	84
7622	7396	7184	6983	6794	6614	6443	6281	6128	5981	5841	85
20	94	82	81	92	12	42	80	26	80	40	86
17	92	80	79	90	10	40	78	24	78	38	87
15	90	78	77	88	09	38	77	23	77	37	88
13	88	76	75	86	07	37	75	22	75	36	89
7610	7386	7174	6974	6784	6605	6435	6274	6120	5974	5834	90
08	83	72	72	82	03	33	72	18	72	33	91
06	81	70	70	81	02	32	70	17	71	32	92
03	79	67	68	79	6600	30	69	16	69	30	93
01	77	65	66	77	6598	28	67	14	68	29	94
7599	7375	7163	6964	6775	6596	6427	6266	6112	5967	5828	95
96	72	61	62	73	95	25	64	11	65	26	96
94	70	59	60	71	93	23	62	10	64	25	97
92	68	57	58	70	91	22	61	08	62	23	98
90	66	55	56	68	89	20	59	06	61	22	99
7587	7364	7153	6954	6766	6588	6418	6258	6105	5960	5821	100

151. Form of Record of Pendulum Observations.

Following is a specimen record of a single swing made with "Apparatus B," belonging to the Coast Survey.

Station: Sawah Loento, Sumatra. Date: May 7, 1901.
Observer: G. L. H. Chronometer: Bond 541 (sid.)

Pendulum *B* 4, Direct, on Knife edge *I*

Observed coincidences	Pressure	Temperature.	Arc.
h m s	mm.	(C).	mm.
D 9 59 03			
U 10 02 12	27 5		
D 05 11	27 5	22°.6	4.5 = 52′
U 08 18	55 0		
D 11 12			
U 14 19			
D 4 54 42			
U 58 12	28 0		
D 5 00 43	28.0	28.8	0 9 = 10′
U 04 08	56 0		
D 06 42			
U 10 06			

Pressure	55.5	Mean temp.	25.70
	4 2	Ther. error	−.30
	51.3 at 0° C.		25°.40

Total interval (mean) $6^h\ 55^m\ 43^s = 24{,}943^s$.
Approximate length of coincidence interval $= 3^m\ 01^s$ 181^s.
Number of coincidence intervals = 138.
Length of one coincidence interval = 180.75.
Period (uncorrected) = 0.5013869.

Uncorrected Period	0.5013869
Corr. for Arc	−5
" " Temp.	−436
" " Press.	+9
" " Rate (No. 541)	+128
" " Flexure	−6
Corrected Period =	0.5013559

152. Calculation of g.

After the period has been corrected for instrumental errors, the value of gravity (g) may be found by comparing the period (P) with that of the same pendulum at some point where the value of g is known, say at Washington. If the value at Washington is g_w, then

$$g = \frac{P_w^2}{P^2} \cdot g_w. \qquad [107]$$

Evidently it is of the greatest importance that the period should not change during a series of observations made for the purpose of comparing P at different stations. The pendulum should be swung at frequent intervals at the base station, to test its invariability; in any case it should be swung at the beginning and end of every series.

Example. Suppose that the mean corrected period of a set of pendulums at a station is 0.5012480, and at Washington, the base station, is 0.5007248, and that g_w is taken as 980.112 dynes. Then, by formula [107], g = 978.067 dynes.

153. Reduction to Sea-Level.

The value of gravity found in the manner just described is the value at the station, assuming the length of the pendulum to be invariable and the chronometer correction to be correct. In comparing values at different stations, however, it is essential to reduce the observed value to the value at sea-level. A formula long used for this purpose is one devised by Bouguer when reducing observations made along the Peruvian arc in 1749. This formula is

$$dg = +\frac{2\,gH}{r}\left(1 - \tfrac{3}{4} \cdot \frac{\delta}{\Delta}\right), \qquad [108]$$

in which H is the elevation of the station above sea-level,

r is the radius of the earth,

δ is the density at the surface,

and Δ is the mean density of the earth.

The first term of this formula allows for the decrease in gravity

due to height alone; the second term, for the increase in attraction due to the topography beneath the station.

The correction for height of station is derived from the law of gravitation, namely that the force of attraction varies inversely as the square of the distance; whence

$$\frac{g_0}{g} = \frac{(r + H)^2}{r^2} = \left(1 + \frac{H}{r}\right)^2 = \left(1 + \frac{2H}{r} \cdot \cdot \cdot\right).$$

Therefore $$g_0 = g\left(1 + \frac{2H}{r}\right). \qquad [109]$$

The correction for topography is based upon the assumption that is due to the attraction of a cylinder whose axis is vertical and whose height is small compared with its width. The attraction on a unit mass at the station is shown by Helmert (*Hohe. Geodäsie*, Vol. II, pp. 142 and 164) to be

$$\Delta g = 2\,\pi k\,\delta H. \qquad (a)$$

The attraction of the sphere on the same mass is

$$g = k\frac{M}{r^2} = k \times \tfrac{4}{8}\,\pi r \Delta. \qquad (b)$$

Dividing (a) by (b) and multiplying by g,

$$\Delta g = g \cdot \tfrac{3}{2} \cdot \frac{\delta}{\Delta} \cdot \frac{H}{r}. \qquad [110]$$

Adding both corrections ([109] and [110]) and remembering that the two are of opposite sign,

$$g_0 = g + g\frac{2H}{r} - g\,\tfrac{3}{2} \cdot \frac{\delta}{\Delta} \cdot \frac{H}{r}$$

$$= g\left(1 + \frac{2H}{r}\left(1 - \tfrac{3}{4} \cdot \frac{\delta}{\Delta}\right)\right).* \qquad [111]$$

Another method of reduction which has been much used is to

* See also Clarke, *Geodesy*, p. 325. For an additional term for irregularity in topography see Coast Survey Report for 1894, p. 22.

omit the last term of Bouguer's formula, and correcting for height only. In this case the correction to g is

$$\text{Corr.} = +\frac{2H}{r}g, \tag{112}$$

or

$$\text{Corr.} = +0.0003086\,H\ \text{(meters)}. \tag{112a}$$

This method was introduced because the former method showed large disagreement between observed and computed values. The second, or "free-air," method showed better agreements, indicating a compensation due to variations of density beneath.

The method employed by Professor Hayford in the Coast Survey investigation shows that still better agreement is obtained by the introduction of the assumption of isostasy. The results corrected by this method show a close general agreement, but in certain localities there is evidence that the isostatic adjustment is imperfect — for example, near Seattle in the United States and at certain places near the Himalayas in India.

154. Calculation of the Compression.

By employing a large number of observed values of g the most probable values of the constants g_e and g_p may be found. From these data the compression may be derived by applying Clairaut's formula,

$$\frac{a-b}{a} = \frac{5}{2}\cdot\frac{c_e}{g_e} - \frac{g_p - g_e}{g_e}. \tag{98}$$

The value of c_e is $\left(\frac{2\pi}{T}\right)^2 \cdot a$, where $T = 86164.09$ seconds and a is the equatorial radius. Using Clarke's value of a the resulting value of c_e is found to be

$$c_e = 0.033916,$$

and using for g_e the value of 978.038,* we obtain

$$\frac{c_e}{g_e} = \frac{1}{288.37} = 0.0034678.$$

* See Coast Survey Special Publication No. 12.

Then for the compression, we have

$$\frac{a-b}{a} = \frac{1}{297.1}.$$

If the more accurate form [98*a*] of Clairaut's equation is employed, the result is

$$\frac{a-b}{a} = \frac{1}{298.2}.$$

By studying a large number of gravity observations in all parts of the world Helmert obtained the value

$$\frac{a-b}{a} = \frac{1}{298.3 \pm 0.7}. \qquad [113]$$

In the publication entitled *Effect of Topography and Isostatic Compensation upon the Intensity of Gravity* the authors give

$$\frac{a-b}{a} = \frac{1}{298.4 \pm 1.5}. \qquad [114]$$

In a later report on gravity work (Coast Survey Special Publication No. 40, 1917), the compression calculated from the observations in the United States, Canada, Europe and India is

$$\frac{a-b}{a} = \frac{1}{297.4}. \qquad [115]$$

By employing Equa. [88] the value of g may be computed for each station on the assumption that the earth is a spheroid. A comparison at each station of the observed and computed values of gravity indicates to what extent the geoid departs from the spheroid at each point.

154a. Gravity Observations at Sea.

Within the past few years gravity observations of a precise character have been made at sea by Dr. Meinesz, of Holland, the measurements being made chiefly in submarines. The apparatus is a modified form of the Stückrath pendulum apparatus, in which two pairs of pendulums oscillate in planes at right angles to each other. The swings are recorded photo-

graphically and time signals are received by radio for determining the periods of oscillation. The results of this work will add greatly to knowledge of the variations of gravity over the ocean areas.

154b. The Eötvös Torsion Balance.

Much research has been carried on in recent years, both of a theoretical and of a practical nature, by means of the Eötvös

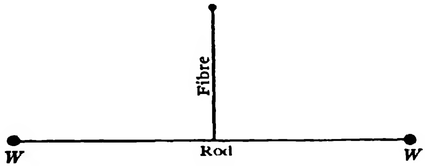

FIG. 122.

torsion balance, a very sensitive instrument which measures minute variations in the force of gravity.

This balance has been designed in two forms. The first may be described as a horizontal rod with nearly all of its weight concentrated at the ends and which is suspended at its center by a fiber from a torsion head. (Fig. 122.)

Under the action of the earth's gravitational field the rod tends to turn into the plane of the prime vertical, because in so doing it "falls" into a position of lower potential.* In order to see how this occurs let us represent on the same diagram the curves of equal potential in the meridian plane and in the prime vertical plane, the former by full lines and the latter by dotted lines. These curves are given consecutive numbers for identification, the same number representing the same equipotential, or level, surface. From Fig. 123 it will be seen that when the rod is in the meridian plane the weighted ends are on surface No. 3. If the rod is turned into the east-west plane the weighted ends are on surface No. 2. A change from the meridian plane to the east-west plane places the end in a position where the potential

* See Art. 165, Chapter X.

is lower. Therefore there is a force tending to turn the rod from the meridian into the prime vertical plane. This force is, of course, very minute.

The torsion of the fiber used in suspending the rod is considerable when compared with the small force in question. In using the balance the position of the torsion head corresponding to

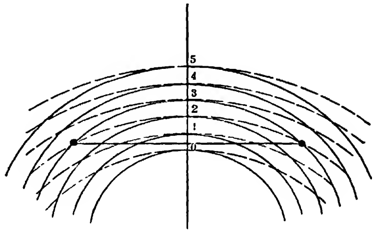

FIG. 123.

"no torsion" is determined. By turning the torsion head in different azimuths, both to the right and to the left of the line of no torsion, the deflection of the rod with reference to the line of no torsion may be measured. From these measurements it is possible to calculate the maximum and the minimum curvature, or $\frac{1}{R_m}$ and $\frac{1}{N}$. For a spheroid these would correspond to the meridian and the prime vertical, but in actual observations they seldom do so, because of local variations in the densities immediately beneath the instrument. The balance also gives us the directions of these two curvatures. It may be noted that these curvatures are related to the second derivatives of the potential function of the gravitational field. This instrument gives numerical values of

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} \quad \text{and of} \quad \frac{\partial^2 V}{\partial x \, \partial y}.$$

The second type of balance deals with forces at two different

levels at the same time. The horizontal rod has at one end a weight which is suspended at some distance below the level of the rod. (Fig. 124.) This balance therefore deals with the same forces which cause a change in the direction of gravity with a change in height. These forces are related to second derivatives of the potential function of the form

$$\frac{\partial^2 V}{\partial x\,\partial z} \quad \text{and} \quad \frac{\partial^2 V}{\partial y\,\partial z}$$

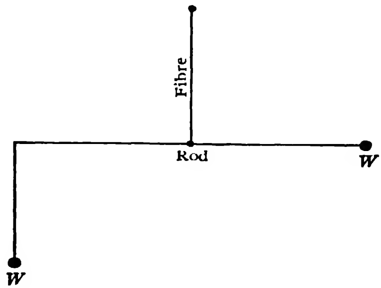

FIG. 124.

and this form of balance gives numerical value of these two derivatives.

The Eötvös balance is extremely sensitive and for this reason can measure very slight variations in the forces acting upon it. It is used to study the geological formation of the earth's crust immediately beneath the instrument station; it has been used successfully for locating salt domes, mineral deposits, etc. of commercial value.

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PROBLEMS

Problem 1. Compute $\frac{a-b}{a}$ from the following data:

Station.	g_0.	Latitude.
		° ′ ″
Umanak, Greenland	982.595	+70 40 29
Sawah Loento, Sumatra.	978 057	−00 41 40

Problem 2. If the coincidence intervals are 5^m during an 8-hour swing, what will be the error in P due to an error of 1^s in noting the time of a coincidence?

Problem 3. If the error in determining the daily rate of the chronometer is 0 .1 what is the error in the period?

CHAPTER X

PRECISE, OR FIRST-ORDER, LEVELING — TRIGONOMETRIC LEVELING

155. Precise Leveling.

The term *precise leveling* (now officially replaced by the term *First Order* leveling) is applied to the operation of determining differences in elevation of successive points on the earth's surface with instruments and methods which, though similar to those used in ordinary leveling, are more refined and capable of yielding a much higher degree of precision. In order to secure the greatest possible accuracy, it is necessary to modify our conception of the nature of a level surface and to introduce certain corrections which are ordinarily negligible. It should be observed that since the line of sight of the instrument is always theoretically perpendicular to the direction of gravity at each station, it lies in a plane which is tangent to the geoid, not to the spheroid. In tracing out a level line by means of the spirit level we are following the curvature of the surface of the geoid.

The term *precise leveling* has for many years been applied to all leveling of a fairly high degree of precision, but there have been various limits of precision prescribed by the different organizations carrying on the work. The accuracy obtainable has been so greatly increased through recent developments in instruments and methods that in 1912 a new class of leveling, known as *leveling of high precision*, was established by the International Geodetic Association; it is to include every line, set of lines, or net, which is run twice in opposite directions, on different dates, and whose errors, both accidental and systematic, computed in accordance with formulas stated in the resolution,* do not exceed

* See Coast Survey Special Publication No. 18, p. 88. See also Report of International Geodetic Association for 1912.

$\pm 1^{mm}$ per kilometer for the probable accidental error and $\pm 0.2^{mm}$ per kilometer for the probable systematic error.

In 1925 the Federal Board of Surveys and Maps adopted the terms *First order*, *second order*, etc., and specified the accuracy for different classes of leveling, as shown in the following table.

LEVELING

First Order.	Second Order.	Third Order.	Fourth Order.
Error of closure of section 0.017 feet $\sqrt{\text{miles}}$ or $4^{mm}\sqrt{\text{kilometers}}$	Error of closure of circuit 0.035 feet $\sqrt{\text{miles}}$ or $8.4^{mm}\sqrt{\text{kilometers}}$	Error of closure of circuit 0.05 feet $\sqrt{\text{miles}}$ or $12^{mm}\sqrt{\text{kilometers}}$	Flying Y levels, vertical angles

Many different instruments have been used in the past for precise leveling, some of the "wye" type and some of the "dumpy" type. All geodetic levels, however, have certain characteristics in common: namely, (1) a telescope of high magnifying power, mounted on a heavy tripod: (2) a sensitive spirit level: (3) a slow-motion screw for centering the bubble: (4) stadia wires for determining the length of sight: and (5) a mirror or other optical device for viewing the bubble from the eye end of the telescope. Before the year 1899 the precise leveling of the United States Coast Survey was done with a wye level and target rods. The target was not set exactly on the level of the instrument, but was set approximately, and corrections to this approximate reading were determined, using the micrometer screw to measure the small vertical angles. Since 1899* a dumpy level of new design has been substituted for the wye level, the self-reading rod adopted, and the micrometer screw used only for centering the bubble. This new instrument and method have been adopted by several other branches of the government service.

* For a discussion of this change in methods see Coast Survey Report for 1899, p. 8, and for a description of the new instrument see Coast Survey Report for 1900, p. 521, and for 1903, p. 200.

156. Instrument.

The geodetic level, sometimes called the *prism level*, is designed to reduce, so far as possible, any errors arising from un-

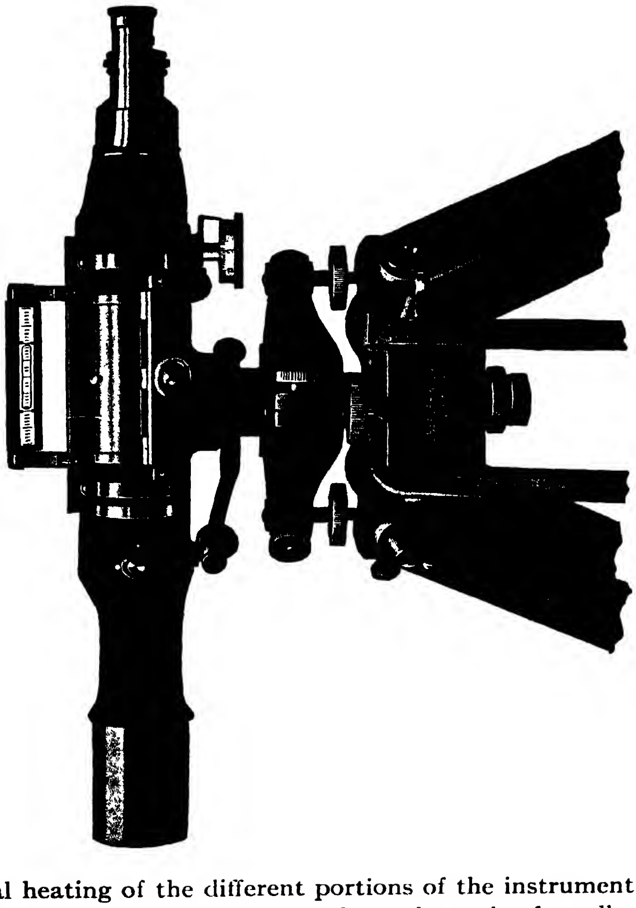


FIG. 125. Geodetic Level. (*C. L. Berger & Sons.*)

equal heating of the different portions of the instrument. (Fig. 125.) The telescope barrel was formerly made of an alloy of iron

and (36%) nickel having a low coefficient of expansion (0.000004 per 1°C.). The most recent levels are of invar having a coefficient of 0.000001. The level vial is set into the telescope tube as low as possible without interfering with the cone of rays from the object glass. This diminishes the effect of differential

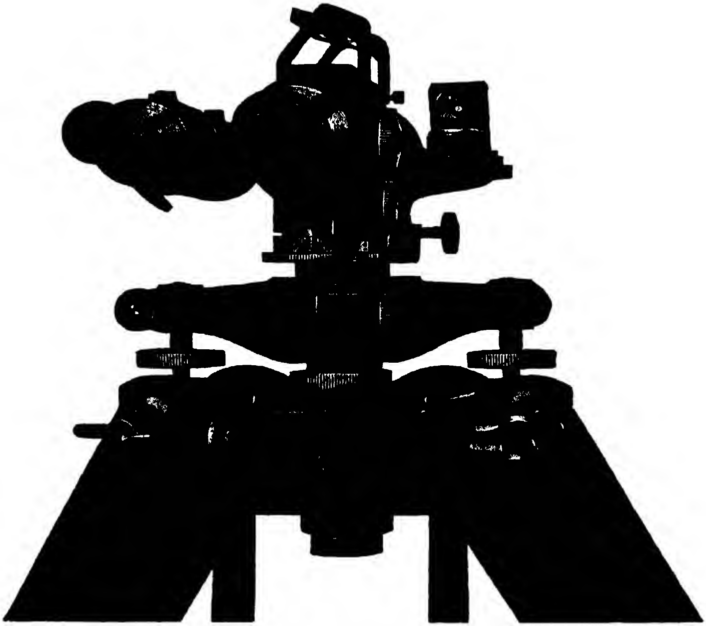


FIG. 126. Geodetic Level. (*C. L. Berger & Sons.*)

expansion of the parts supporting the level. The level vial is provided with an air chamber for varying the length of bubble. At one side of the telescope is another (similar) tube containing a pair of prisms which, together with a mirror mounted above the telescope, enable the observer to view the ends of the bubble with the left eye at the same time that he looks at the rod with the right eye. The arrangement of mirror and prisms is

such that there is no parallax caused by the glass in the level or the mirror. The instrument is provided with the usual small level for the approximate leveling of the base. It is mounted on an unusually high tripod so as to avoid refraction near the surface of the ground. The magnifying power of the telescope is about 43 diameters; the level division ($\frac{1}{10}$ inch) has an angular value of about $1''.7$. The stadia constant varies from 1 in 328 to 1 in 348 in different levels.

In the instrument shown in Fig. 125 the pivot, about which the telescope is tilted by the motion of the micrometer screw, is placed near the forward end. In the recent levels the pivot is at the center.

157. Leveling Rods.

The rod in use at present (1929) by the Coast & Geodetic Survey is of one piece about 3.1 m. long, flat in section (see Fig. 127) and has a strip of invar attached to the front face, on which the centimeter divisions are painted. The details of the rod are shown in the cut. The rod is provided with a centigrade thermometer, and with a circular spirit level for plumbing the rod. Many rods have a scale of feet (or meters) painted on the back for rough check readings.

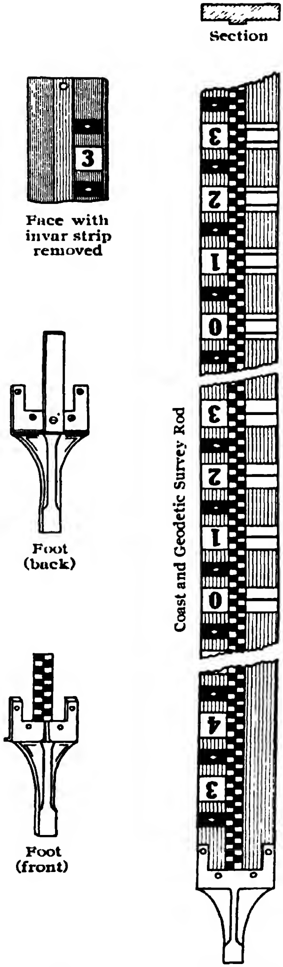


FIG. 127. Rod for Geodetic Leveling

The rod used by the U. S. Geological Survey is similar in general de-

sign, but is divided into hundredths of yards. This is convenient when obtaining elevations in feet, since the sum of the three thread readings gives at once the mean reading in feet.

158. Turning Points.

Special steel pins are usually carried by leveling parties for use on highways or when the usual turning points are not available. Since most of the first-order leveling is carried along railway lines the regular turning point is either a nail driven into a tie or a rail spike near the middle of a rail. The top of the rail was formerly used but this practice has been discontinued, as it was suspected that cumulative errors from this source may have entered the results.

159. Adjustments.

The adjustments of the level are nearly the same as those of the ordinary dumpy level. The rough level is adjusted so as to remain in the center when the telescope is revolved about the vertical axis. The axis of the long bubble tube is adjusted parallel to the line of sight of the telescope whenever it is appreciably in error. This adjustment is tested each day by taking four readings, like those used in the "peg" method, except that the shorter sights are about 10 meters in length and the longer sights are of the usual length, say 100^m. From these four readings a factor C is computed, which is the ratio of the correction for any reading to the corresponding rod interval. The difference in the sums of the foresight and backsight thread intervals at any set-up is to be multiplied by this factor C.

To find an expression for C, call n_1 and n_2 the rod readings for the nearer sights, and d_1 and d_2 the rod readings for the distant sights, s_1 and s_2 the near stadia intervals, and S_1 and S_2 the distant stadia intervals, the subscripts referring to the first and second instrument positions. Then the true difference in elevation from the first set-up is

$$(n_1 + Cs_1) - (d_1 + CS_1),$$

and for the second set-up,

$$(d_2 + CS_2) - (n_2 + Cs_2).$$

Equating and solving for C we have,

$$C = \frac{(n_1 + n_2) - (d_1 + d_2)}{(S_1 + S_2) - (s_1 + s_2)}. \qquad [116]$$

C is + if the line of sight is inclined downward.

Below is a table showing a determination of C (from Coast Survey Report for 1903)

DETERMINATION OF *C*. 8.20 A.M., AUGUST 28, 1900

(Left-hand page.)				(Right-hand page.)			
Number of station.	Thread reading, backsight.	Mean.	Thread interval	Rod	Thread reading, foresight	Mean.	Thread interval.
A	1515 1528 1542	1528 3	13 14 27	W	0357 0462 0566	0461 7	105 104 209
B	2252 2357 2462	2357 0	105 105 210	W	1276 1288 1301	1288 3	12 13 25
		0461 7	419			1528 3	
		2818 7	52			2816 6	
Corr. for curv. and ref.		−0 8	367			2817 9	
		2817.9			367) −1 3 (−0.004 = *C*		

In this table A and B refer to the instrument stations, not to the points where the rods are held. The three numbers given in the column "thread interval" are first, the space between the lower and middle threads, second, the space between the middle and upper threads, and third, the sum of these two. The mean for the three rod readings (thread readings, second column) is found by dividing the difference between the upper and lower thread intervals by 3 and applying the result as a correction to the middle thread reading. The result is placed in the third column. For example, the difference between 13 and 14 divided by 3 gives 0.3 mm., which added to 1528 gives 1528.3, the correct mean of the three readings.

In order to make the additions conveniently when solving for C, the means are carried across to the opposite pages, that is the 1528.3 is carried to the right-hand side and the 0461.7 to the left-hand side.

The distant rod readings are corrected for the effect of curvature and refraction, which may be conveniently taken from Table H. In the example the distances are not corrected separately, but the sum of the two corrections is subtracted from the sum of the distant readings. In this particular instrument the stadia factor is 348, that is, the distance equals 348 times the thread interval. The distance corresponding to 0.209 m. or 0.210 m. is a little over 73 m. In Table H this gives a correction of 0.4 mm. for each distant reading. The sum of the two, or 0.8 mm., is subtracted from 2818.7 mm. The value of the constant is therefore

$$C = \frac{2816.6 - 2817.9}{419 - 52} = \frac{-1.3}{367} = -.004$$

This indicates that the level is in good adjustment. The negative sign shows that the line of sight is inclined upward and that all rod readings are too great by the amount Cs.

If the instrument were to be adjusted, it would be done by sighting at a distant rod with the bubble in the center of the tube and taking a reading. The telescope is then inclined upward if C is $+$ (lowered if C is $-$) until the reading is increased (or decreased) by an amount equal to Cs. With the telescope in this position the bubble is centered by means of an adjustment screw on the level case inside the telescope barrel.

After the instrument is adjusted a new value of C must be determined to be used in correcting the results of subsequent leveling.

If the value of C is less than 0.005, the instrument should not be adjusted. If between 0.005 and 0.010, the observer is advised not to adjust. If over 0.010, the adjustment should be made. The adjustment is made by moving the level rather than the

cross-hair ring, in order to avoid moving the line of sight away from the optical axis.

The rod level should be adjusted by using a plumb line to bring the rod to a truly vertical position and then centering the bubble by the adjusting screws. If the plumb line can be fastened temporarily to a corner of the top of the rod and the string is long enough to allow the plumb bob to swing clear, the rod may be plumbed in two directions at the same time.

160. Method of Observing.*

It is customary to use two rods, the one that is held for a foresight on a certain turning point being kept at the same turning point for a back sight. This results in alternating the foresights and backsights; that is, if the backsight is taken first at the first instrument station, the foresight is taken first at the second instrument station, and so on throughout the section. This is done so that any settlement of instrument or rod will be eliminated as completely as possible. Suppose that during the interval between a backsight reading and a foresight reading the instrument settles one millimeter. In this case the recorded backsight reading is too high for the position of the instrument at the time of the foresight reading, and the turning point is assigned too high an elevation. At the following station the foresight is taken first. The settlement of the level results in the backsight reading being 1^{mm} too small and in the turning point elevation being that amount too low, which just balances the previous error.

At the first station the instrument is set up and leveled,† and all three hairs are read on the back rod, the level being kept central at each reading. As soon as possible thereafter the three hairs are read in a similar manner on the forward rod. The readings are estimated to millimeters. The level should be shaded from

* The General Instructions for First Order Leveling will be found in Coast Survey Special Publication No. 140.

† The observer should determine the micrometer reading corresponding to the "reversing point" of the level, and set the micrometer at this reading when leveling the instrument

the sun in order to avoid unequal heating of its parts. In selecting instrument and rod points, the observer must keep the difference in length of the forward and backward sight less than 10 meters on any one set-up and less than 20 meters for the accumulated difference at any time. The readings of the upper and lower (stadia) wires enable the recorder to determine the differ-

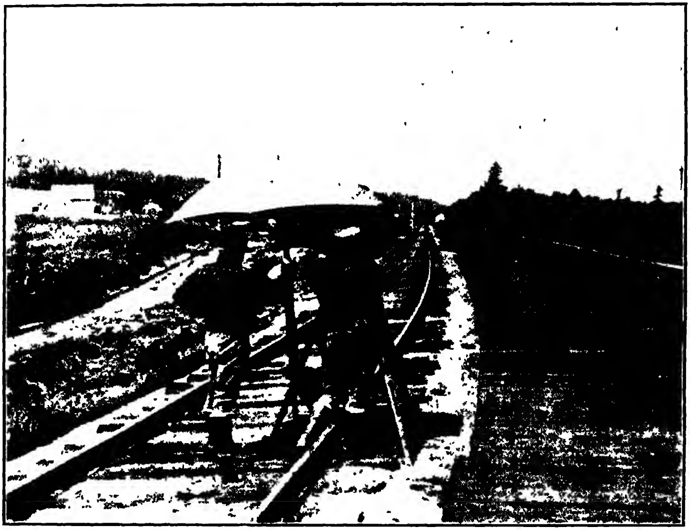


FIG. 128. First-order Leveling Party.

ence in distance at each set-up. When leveling along railway tracks the lengths of sights are equalized approximately by counting rails. The maximum length of sight allowable is 150^m, a distance reached only under exceptionally favorable conditions. As already stated, at odd-numbered stations the backsight is taken first; at even-numbered stations the foresight is taken first. This results in the same rod being read first each time. The rod thermometers are read at each set-up.

If there is much irregular atmospheric refraction near the

ground this will result in the rapid variations in the reading of the upper thread. This should be avoided so far as possible by setting the instrument high above the ground and by avoiding turning points which bring the line near the surface. It is found that whenever there are variations in the reading on a thread the accuracy is increased by selecting the lowest (least) reading rather than the mean.

Lines between bench marks are divided into sections of about one kilometer each. Each of these sections is run forward and backward. If the two differences in elevation so determined are found to differ by more than $4^{mm}\sqrt{K}$ (K = kilometers), both runnings must be repeated until such a check is obtained. Lines may be run with such care that it is seldom necessary to repeat, but the maximum economy appears to be reached when from 5 to 15 per cent of the sections have to be re-run.

On page 312 is a set of notes used in leveling with this instrument (Coast Survey Report, 1903). The instrument stations (not rod stations) are numbered in the first column. The thread readings, means, and thread intervals are arranged exactly as in the preceding table (p. 307). The last column "Sum of intervals" is seen to contain in each case the sum of all the preceding thread intervals. It is by means of this column that the recorder is able to see if the foresights are too long or too short and to warn the rodman that the foresight should be made shorter or longer in order to balance the two columns. One millimeter of stadia interval corresponds nearly to one foot of distance, so the recorder can at once call out the number of feet in this correction.

The records are sometimes kept by entering the readings directly on an adding machine, one bank of keys being used for the backsight, the other bank for the foresight.

The geodetic level is sometimes mounted on a motor velocipede car, either over its center or at one side so as to be read while the observer is on the ground. A similar car is used for the transportation of the recorder and the two rodmen.

When a river crossing is being made the method is modified as

follows: Targets are placed on the rod so that one is below and one above the level line. The middle hair is made to bisect each target in turn by moving the micrometer screw. The readings of the micrometer drum for the two targets and for the zero position of the bubble, combined with the readings of the targets on the rod, furnish sufficient data for computing the rod reading of the middle hair for a horizontal sight. Observations are made simultaneously with two instruments. Observers then exchange places and repeat the whole operation.

SPIRIT LEVELING

(Left-hand page.) Date: August 29, 1903. Sun: C. Forward. ~~Backward~~ (Strike out one word.)

(Right-hand page.) From B.M. 68. To B.M. · G. Wind: S.T.

No. of station.	Thread reading, back-sight.	Mean.	Thread interval.	Sum of intervals.	Rod and temp.	Thread reading fore-sight.	Mean.	Thread interval.	Sum of intervals.
43	0674		99		V	2683		99	
	0773	0773.0	99		38	2782	2782.3	100	
	0872		198			2882		199	
	0925		106		W	2415		103	
44	1031	1030.3	104		35	2518	2518.0	103	
	1135		210	408		2621		206	405
	0484		98		V	2510		96	
45	0582	0582.3	99		35	2606	2606.0	96	
	0681		197	605		2702		192	597
	0398		97		W	2859		95	
46	0495	0495.0	97		34	2955	2954.7	95	
	0592		194	799		3050		191	788
	1027		26		V	1006		29	
47	1053	1053.3	27		34	1035	1034.7	28	
	1080		53	852		1063		57	845
							11895.7		
		3933.9					−7961.8		
							2 : 25 P.M.		

161. Computing the Results.

In computing the difference in elevation the sums of the backsights and the foresights are computed and checked. In the notes shown these are 3933.9 and 11895.7 respectively. The difference, −7961.8, is the uncorrected difference in elevation, and shows that B.M. *G* is lower than B.M. 68. The notes show that the sum of the backsight distances (852) is but 7^{mm} greater than the sum of the foresight distances (845); this means that the sights balance within 2.4 m. The correction for non-adjustment of level is but $7 \times .004 = 0.03$ mm. This correction is ordinarily negligible if the balancing has been properly attended to.

The difference of elevation as found from the notes must be corrected for (1) non-adjustment of level (already mentioned), (2) curvature and refraction (Table I, p. 316), (3) error in length of rod, (usually determined at Bureau of Standards), (4) temperature of rod if correction is appreciable, and (5) for convergence of level surface, or the "orthometric correction," (see Art. 169).

162. Bench Marks.

The bench marks used in geodetic leveling are of various types. Wherever it is practicable, the metallic plates shown in Fig. 129, set in concrete posts, are used to mark the points, but nearly all of the kinds of bench marks which are used by engineers are used also in this class of work. The distance between benches is not allowed to exceed 15 kilometers; every 100 kilometer section should have at least 20 bench marks, a good average distance being 2.5 kilometers. In cities the old bench marks are often utilized for the first-order levels.

163. Sources of Error.

The sources of error which it is particularly necessary to study in this class of work are (1) unequal effects of temperature changes on the adjustment of the instrument, (2) gradual rising or settling of the instrument or rods, (3) variations in refraction of the air in different parts of the day and on different days, (4) unequal lengths of sights, (5) errors in length and temperature of rod, and (6) convergence of level surfaces.

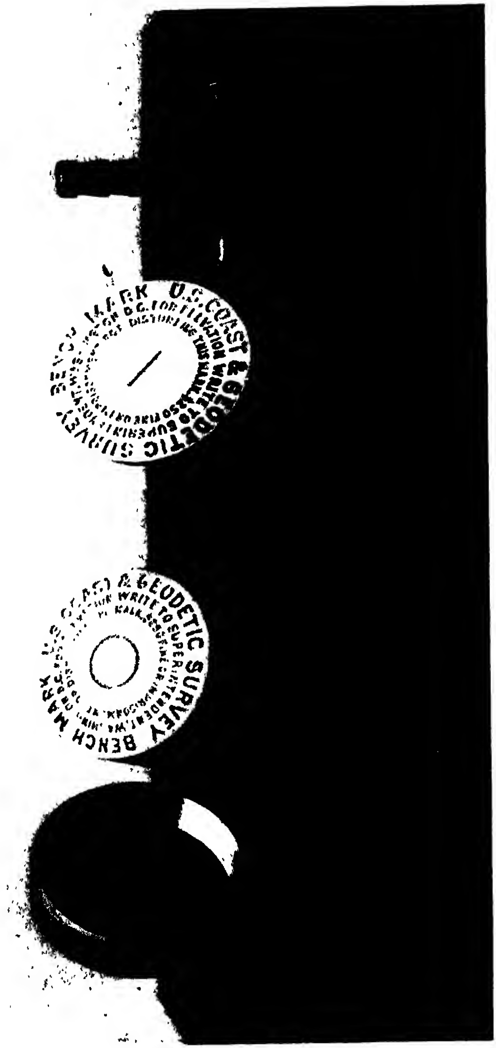


FIG. 120. Bench Marks. *(Coast and Geodetic Survey.)*

TABLE H. — TOTAL CORRECTION FOR CURVATURE AND REFRACTION

Distance.	Correction to rod reading.	Distance.	Correction to rod reading.
m. *m.*	*mm.*	*m.*	*mm.*
0 to 27	0 0	160	−1 8
28 to 47	−0 1	170	−2 1
48 to 60	−0.2	180	−2 3
61 to 72	−0 3	190	−2 6
73 to 81	−0.4	200	−2 8
82 to 90	−0 5	210	−3 0
91 to 98	−0 6	220	−3 3
99 to 105	−0 7	230	−3 7
106 to 112	−0 8	240	−4 0
113 to 118	−0 9	250	−4 3
119 to 124	−1 0	260	−4 7
125 to 130	−1.1	270	−5.0
131 to 136	−1.2	280	−5 4
137 to 141	−1 3	290	−5 8
142 to 146	−1 4	300	−6 2
147 to 150	−1 5		

164. Datum.

The datum for geodetic levels is mean sea-level, or the surface of the geoid, as found from tidal observations. This is assumed to be correctly given by the mean of the several "annual means" as derived from tidal observations for sea-level. The heights of the tide are recorded on an automatic gauge. (See Fig. 130.) The vertical motion of the float is reduced (the ratio depending upon the range of tide) by passing the connecting wire and cord over a series of pulleys, and is communicated to a recording pencil which marks on a sheet of paper passing over a revolving drum. The drum is revolved at a uniform rate by clock mechanism. The height of the water is referred to a bench mark in the vicinity. Figure 131 shows a smaller (portable) gauge also used by the Coast and Geodetic Survey.

Observations of the tide should be extended over a period of at least one year in order to determine sea-level with sufficient precision for this class of leveling. In the tidal records at some stations there appear to be small systematic variations in the

TABLE I. — DIFFERENTIAL CORRECTION FOR CURVATURE AND REFRACTION

Mean length of sight in rod interval in millimeters	Difference of sights in rod interval in millimeters.																											
	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50	52	54	56	58
10	0	0	0	0	0																								
20	0	0	0	0	0	0	0	0	0	0																			
30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0														
40	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0									
50	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0				
60	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
70	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
80	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1
90	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1
100	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1
110	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
120	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
130	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
140	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
150	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
160	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
170	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2
180	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2
190	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2
200	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2
210	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2
220	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2
230	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2
240	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2
250	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2
260	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2
270	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2
280	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	2
290	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	2	3
300	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	3	3
310	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	3	3	3
320	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	3	3	3	3
330	0	0	0	0	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	3	3	3	3	3
340	0	0	0	0	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	3	3	3	3	3
350	0	0	0	0	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	3	3	3	3	3	3
360	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	3	3	3	3	3	3	3
400	0	0	0	0	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	3	3	3	3	3	3	3	3	4
440	0	0	0	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	3	3	3	3	3	3	3	3	4	4	4
480	0	0	0	1	1	1	1	1	1	1	2	2	2	2	2	2	2	3	3	3	3	3	3	4	4	4	4	4	4
520	0	0	1	1	1	1	1	1	1	2	2	2	2	2	2	3	3	3	3	3	3	3	4	4	4	4	4	4	4

annual means extending over periods of several years; but, taking the records as a whole, the variations do not seem to follow any particular law, and they have been treated as accidental.

From time to time evidence has appeared which indicated that lines of levels were not following exactly parallel to the surface shown by the tidal observations, and that mean sea level

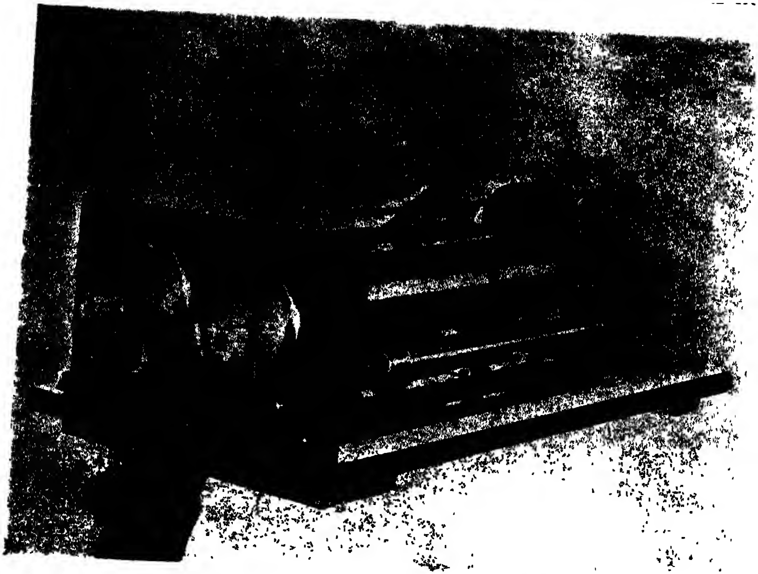


FIG. 130. Automatic Tide Gauge. (*Coast and Geodetic Survey.*)

everywhere at the same elevation, under the ordinary definition of elevation. In other words, the assumption that mean sea-level is an equipotential surface* was not exactly borne out by the tidal observations and the leveling operations.

In 1928 the Coast and Geodetic Survey made a readjustment of 50 000 miles of first-order leveling, holding fixed only a single mean sea-level station. The result showed that the Pacific

* See Art. 168.

Ocean on our western coast is about 2 feet higher (average) than the water of the Atlantic and Gulf on the eastern and southern coasts. The results also show that on both the Atlantic and the Pacific Oceans the mean sea-level slopes upward to the north.

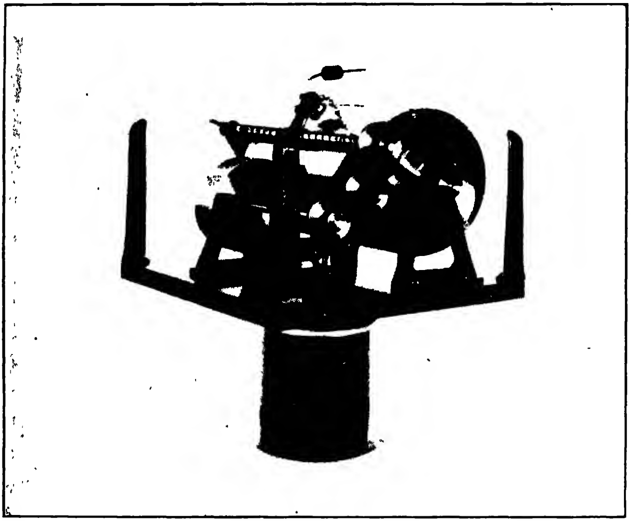


FIG. 131. Portable Automatic Tide Gauge. (*Coast and Geodetic Survey*)

165. Potential.

In order to investigate the nature of the *orthometric correction* (mentioned in Art. 161) due to the convergence of level surfaces, it will be necessary to consider first some of the mechanical principles of the earth's gravitation and rotation.

Whenever two attracting bodies are separated, work is done upon them and energy is stored up; that is, the potential energy of the system is increased. The change in potential energy is

measured by the amount of work done. That is, the potential energy stored up is proportional to the height. When the bodies are an infinite distance apart, the potential energy is a maximum; when the bodies are in contact, the potential energy of the system is zero. If the masses are free to move, they will always move in such a direction as to *diminish* the potential energy of the system.

If we imagine a unit mass placed at any point P in space and attracted by a mass M, and if the potential energy of the unit mass be measured by the work done upon it to move it from P to infinity, this quantity of potential energy is the property of the given point P. It is called the *potential* at that point. It is not necessary that there should actually be a unit mass at the point, but the conditions are such that if a unit mass were placed at P, it would have this amount of potential energy. *Potential* is analogous to *level*. A point at a high level has a high potential.

166. The Potential Function due to Attraction.

If an attracting body M be divided into small elements, and the mass Δm of each element be divided by its distance from a point P, the limits of the sum of all these fractions as the elements are made smaller, is called the value at P of the potential function due to M, or simply the potential of P. This function becomes numerically smaller as the height increases. It will be seen later that the gravitational potential is negative. Calling this function V, then

$$V = \lim_{m=0} \sum \frac{\Delta m}{r} \qquad [117]$$

or, if Δm is of density δ and has the coördinates x', y', z', and P has the coördinates x, y, z, then

$$V = \int\int\int \frac{\delta \cdot dx' \cdot dy' \cdot dz'}{[(x'-x)^2 + (y'-y)^2 + (z'-z)^2]^{\frac{1}{2}}}. \qquad [118]$$

The integration over the entire mass gives the value of the potential function at P.*

* See Peirce, *Theory of the Newtonian Potential Function.*

167. The Potential Function as a Measure of Work Done. The amount of work required to move a unit mass (concentrated at a point) from a point P_1 to another point P_2, by any path (Fig. 132), against the attraction of a mass M (concentrated at its center), is equal to the change in potential, $V_1 - V_2$, where V_1 and V_2 are the values of the potential function at the points P_1 and P_2. To show this, let r_1 and r_2 be the distances from the

FIG. 132

center of M to the points P_1 and P_2. The attraction of M on a unit mass at P_1 equals

$$k\frac{M}{r^2},$$

k being a constant whose value depends upon the units employed.

If we take as the unit of attraction the "attraction of a unit mass on a unit mass a units' distance away" then the attraction equals

$$\frac{M}{r^2}$$

The work done in moving the unit mass through a small space dr from P_1 toward P_2 equals (force times space)

$$\frac{M}{r^2}\cdot dr$$

The work done in moving it from P_1 to P_2 is

$$\int_{r_1}^{r_2}\frac{M}{r_2}\cdot dr$$

$$= M\left[-\frac{1}{r}\right]_{r_1}^{r_2}$$

$$\frac{M}{r_1} \quad \frac{M}{r_2}$$

$$= V_1 - V_2. \qquad [119]$$

That is, the work done equals the change in potential.

If the point P_2 be supposed at infinity, V_2 becomes zero, and the potential at P_1 then equals the work done in moving the unit mass from P_1 to infinity; or, it is the work done by it in moving from infinity to the point P_1.

168. Equipotential Surfaces.

A level surface, or equipotential surface, is one having at every point the same gravity potential. It is everywhere perpendicular to the direction of gravity.*

The mean surface of the ocean is such a surface. The surface of any lake is also an equipotential surface. Such a surface is also a surface of equal hydrostatic pressure.

The equation of an equipotential surface may be written as

$$\frac{\partial V}{\partial s} = 0$$

or

$$V = \text{constant}.$$

If we consider the centrifugal force as well as the attractive force, and call the potential W, then it may be shown† that

$$W = \int \frac{dm}{r} + \tfrac{1}{2}\,(x'^2 + y'^2)\,\omega^2.$$

For this case also we shall find that the work done equals the change in potential. In this case too we have for the equation of a level surface

$$\frac{\partial W}{\partial s} = 0$$

or

$$W = \text{constant}.$$

From these equations it follows that

$$g = -\frac{\partial W}{\partial h}$$

and

$$dW = -g\,dh.$$

* It may be proved that if there is a resultant force at any point in space due to attracting masses, this force acts in the direction of the normal to the equipotential surface through the point (see Peirce, *Theory of the Newtonian Potential Function*, p. 38). It should be kept in mind that "the force of gravity" is the resultant of the force of attraction and the centrifugal force.

† Helmert, *Höheren* Geodäsie, Vol. II, p. 8.

From the preceding discussion it is evident that if we consider any two equipotential surfaces, the difference in potential is the work done upon a unit mass in moving it from one surface to the other. The difference in potential is independent of any particular points on those surfaces and of the path followed in passing from one to the other; for example, the work done in raising a unit mass from sea-level to the south end of a lake is the same as the work done in raising a unit mass from sea-level to the north end of the lake. Since the work done is the force (w) times the distance (dh) through which it acts, it is clear that $-w \times dh$ is a constant between two nearby level surfaces. Also, since $w = mg$, g varies as the weight (force), and therefore $-g \times dh$ is a constant between these two level surfaces.

The force of gravity is less at the equator (Art. 144) than it is at the poles, chiefly on account of the action of the centrifugal force. At the equator we have $g_e = 978$ and at the pole, $g_p =$

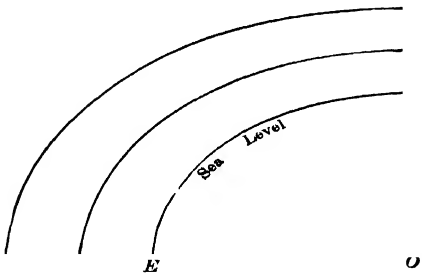


FIG. 133.

983, nearly. Hence we should expect to find that a given level surface is farther from sea-level at the equator than it is at a point nearer the pole. If several such surfaces be drawn (Fig. 133), they will be seen to converge toward the pole. They are parallel to each other at the equator and at the poles, and have their greatest difference in direction at latitude 45°.

Since g is about one-half of one per cent less at the equator than at the pole, the height h between surfaces is about one-half of one

per cent greater at the equator. Hence, if a level surface were 1000 meters above the sea-surface at the equator, it would be only 995 meters above sea-level at the pole. A surface at half the elevation would converge (very nearly) half as much. In the line of levels run from San Diego to Seattle the convergence was found to be about $1\frac{1}{4}$ meters, showing that at high elevations this error is by no means a negligible one in first-order leveling.

It is evident that if a series of bench marks is established along a meridian (in the northern hemisphere), and all are placed at the same elevation, using the ordinary methods, those at the northern end of the line lie nearer to sea-level than those at the southern end of the line. It becomes necessary, then, to revise the definition of *elevation*.

If the ordinary definition of elevation is retained, and no allowance made for convergence of level surfaces, then different results

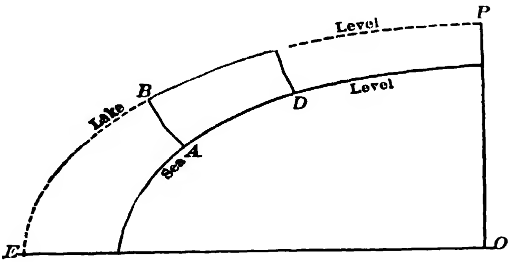


FIG 134.

for the elevation of a point will be obtained, according to which path is followed. If we measure vertically upward from A to B (Fig. 134), and then level by means of the water surface BC, we obtain a greater height for point C than we should if we leveled by water from A to D and then measured vertically upward from D to C. If a correction is applied, however, to allow for the convergences of these surfaces, the result is that different portions of the lake surface have different elevations, which is apparently absurd if the true nature of the level surface is not understood.

In order to avoid this apparent difficulty another method sometimes employed is to number all the surfaces with a serial number (called the Dynamic Number), so that all points on the same surface will have their elevation expressed by the same number. This number is defined as the work required to raise one kilogram from sea-level to the given surface, the unit being the kilogram-meter at sea-level in latitude 45°. The United States Coast Survey has adopted the method of applying to ordinary elevations the correction for convergence, called the *O thometric Correction.* The Standard Elevations of the Coast Survey in Special Publication No. 18 are given by the *Orthometric Elevation.*

169. The Orthometric Correction.

Let W be the work (in absolute units) required to raise a unit mass from sea-level to a point at orthometric elevation h, and let H be the dynamic number of the surface through the point, defined by the quotient $W \div g_{45}$, where g_{45} is the value of g at sea-level in latitude 45° (Equa. [96*a*], p. 270). Then, since $g \times dh$ is constant for two level surfaces separated by height dh,

$$W = \int_0^h g\, dh = g_{45} \int_0^h (1 - 0.002644 \cos 2\phi \ldots)\, dh$$

in which the integration takes place along the curved vertical.

$$\text{Integrating, } W = g_{45}\Big[(1 - 0.002644 \cos 2\phi)\, h \ldots\Big]_0^h. \qquad [120]$$

The dynamic number

$$= H = \frac{W}{g_{45}} = h\,(1 - 0.002644 \cos 2\phi \ldots). \qquad [121]$$

The dynamic number may be computed from Equa. [121] if we neglect local variations in g and consider the earth to be truly spheroidal. It will be observed that H is constant for any surface, but that h is not. The two elevations, H and h, are equal in latitude 45°.

To find the correction to the elevation due to a change in the latitude, differentiate the last equation with respect to ϕ as the

independent variable, and we obtain

$$0 = dh\,(1 - 0.002644 \cos 2\,\phi) + 0.005288\, h \sin 2\,\phi\, d\phi,$$

and
$$dh = -\frac{0.005288\, h \sin 2\,\phi\, d\phi}{1 - 0.002644 \cos 2\,\phi} \qquad [122]$$

$$= -(0.005288\, h \sin 2\,\phi)(1 + 0.002644 \cos 2\,\phi \ldots)\, d\phi \text{ arc } 1',^{*} \qquad [123]$$

the factor arc $1'$ being introduced to reduce $d\phi$ to minutes of arc.

A more definite idea of the magnitude of this correction may be gained from the following example. Assuming that the elevation of Lake Michigan is 177 meters at Chicago, latitude $41^\circ\,53'$, what is the elevation of the lake at Milwaukee, in latitude $43^\circ\,03'$? In the formula, $h = 177^m$, $d\phi = 70'$, and $\phi = 42^\circ\,28'$; the computed values of dh is -0.0190^m, and the lake level at Milwaukee is therefore 176.9810 meters. Tables for computing the orthometric correction will be found in Coast Survey Special Publication No. 140.

The relation between the dynamic numbers and the orthometric elevations is illustrated in the following table, which is an extract from the special publication just mentioned.

Station.	Latitude.	Orth. elev meters.	Dyn number.
Smithland, La.	30 55	14 7729	14 7545
Meridian, Miss.	32 22	104 9494	104 8292
Amblersburg, W. Va.	39 23	494 9221	494 6287
Summit, Cal.	34 20	1165 4345	1164 1008
Riordan, Ariz.	35 13	2216 5452	2213 8112

170. The Curved Vertical.

In view of what has been said regarding the change in the direction of level surfaces with an increase in elevation, it is clear that the vertical line is curved, being concave toward the pole, and therefore that any observation for latitude made at a point

* For additional terms, neglected in the above formula, see Coast and Geodetic Survey Special Publication No. 18, p. 49. See also Ch. Lallemand, *Nivellement de Haute Precision, Encyclopédie des Travaux Publics, Paris,* 1912.

above sea-level is referred, not to the true normal to the surface at sea-level, but to the direction of that portion of the vertical which is at the elevation (h) of the station. In order to determine the amount of the correction to reduce the observed latitude to its value at sea-level, refer again to Equa. [122], p. 325. An inspection will show that the denominator of this fraction is usually not far from unity; and since the correction desired is itself quite small, we may assume

$$dh = -0.005288\,h \sin 2\,\phi\,d\phi. \qquad [124]$$

The correction to the observed latitude is the difference in the slope of the two surfaces (sea-level and the level of the station) measured in the plane of the meridian. From Fig. 135 it is seen

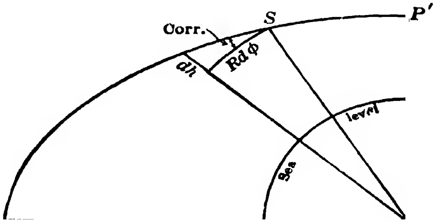


Fig. 135.

that the angle between the level surface through S and a surface parallel to sea-level drawn through S is $dh \div R\,d\phi$. But, by Equa. [124],

$$\frac{dh}{R\,d\phi} = -\frac{0.005288\,h \sin 2\,\phi}{R}.$$

Reducing this to seconds of arc,

$$\frac{dh}{R\,d\phi} \qquad \frac{0.005288\,h \sin 2\,\phi}{R \text{ arc } 1'}$$

Since R arc $1'' = 101.3$ feet (very nearly), the correction to the latitude may be written

$$-\,0''.0522\,h \sin 2\,\phi, \qquad [125]$$

where h is in *thousands of feet;* or, if h is in meters, the correction is

$$- 0.000171\, h \sin 2\phi. \qquad [126]$$

Values of this correction will be found in Table VII, p. 416.

171. Trigonometric Leveling.

The method of measuring the vertical angles between triangulation stations has already been described in the chapter on field-work. From the field note-book we have the several measures of the angles, the height of the instrument, and also of the point sighted in each case above the station marks. The elevation of one station above sea-level is assumed to be known, and that of the other is to be computed. Before this can be done, the angle must be reduced to the value it would have if the instrument and the point sighted were coincident with the station marks.

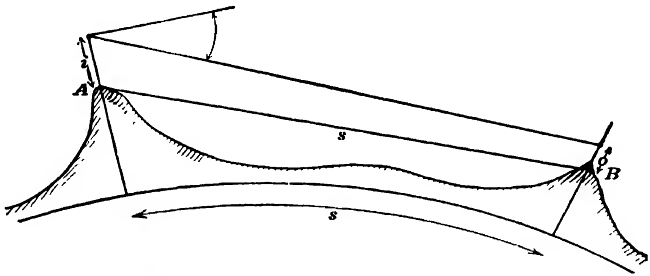


FIG. 136. Height of instrument and object, trigonometric leveling.

172. Reduction to Station Mark.

From the diagram (Fig. 136) it is evident that if i is the height of the instrument at A, and o that of the object sighted at B, and s the distance between stations, obtained from the triangulation, then the correction to the vertical angle at A is

$$\text{Corr.} = \frac{i - o}{s \text{ arc } 1''} \qquad [127]$$

Four places in the logarithms are sufficient in computing this correction.

This reduction need be made only in case of reciprocal obser-

vations, that is, observations of the vertical angle from both ends of the line. In case of observations from one station only, the quantity $i - o$, in meters, can be applied directly to the computed difference in elevation.

When a sight is taken from one station P_1 to another station P_2, the verticals of the two stations do not (in general) intersect, because they lie in different planes. If we imagine a plane which

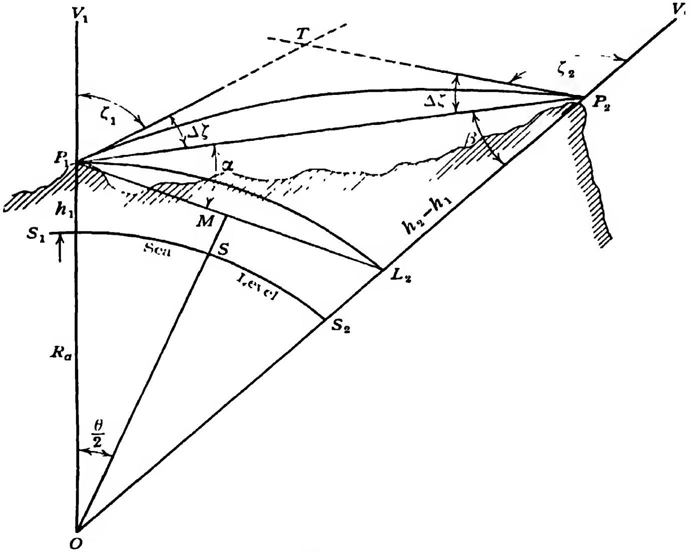


FIG. 137.

is parallel to both verticals, and then project both verticals onto this plane, we obtain the result shown in Fig. 137.

173. Reciprocal Observations of Zenith Distances.

In Fig. 137, P_1 and P_2 represent the two instrument stations; their elevations above sea-level are $P_1S_1 = h_1$ and $P_2S_2 = h_2$. The ray of light is assumed to take the form of a circular curve, whose radius is determined by the coefficient of refraction used in the calculation. The two measured zenith distances are ζ_1 and ζ_2.

The angle of refraction is $\Delta\zeta = TP_1P_2 = TP_2P_1 = m\theta$, where m is the coefficient of refraction, and θ the central angle P_1OP_2. The radius of curvature of the section S_1S_2 is $R\alpha$, approximately equal to OS_1, or to OS_2.

The quantity to be computed is the difference in elevation $h_2 - h_1$, which may be found by solving the triangle $P_1P_2L_2$.*

In the triangle $P_1P_2L_2$, $P_2L_2 = h_2 - h_1$, the desired difference in elevation; P_1L_2 is the chord joining the two verticals at the level surface through P_1. Observing that $P_1M = (R_\alpha + h_1) \sin\frac{\theta}{2}$, and $P_1L_2 = 2(R_\alpha + h_1)\sin\frac{\theta}{2}$, we have, by applying the law of sines,

$$h_2 - h_1 = 2(R_\alpha + h_1)\sin\frac{\theta}{2} \cdot \frac{\sin\alpha}{\sin\beta}. \tag{e}$$

But in the triangle $P_1L_2P_2$

$$\begin{aligned}\alpha &= \left(90^\circ - \frac{\theta}{2}\right) - \beta \\ &= \left(90^\circ - \frac{\theta}{2}\right) - (180^\circ - \zeta_2 - \Delta\zeta) \\ &= -90^\circ - \frac{\theta}{2} + \zeta_2 + \Delta\zeta. \qquad (f)\end{aligned}$$

Also
$$\begin{aligned}\alpha &= 180^\circ - \left[\zeta_1 + \Delta\zeta + 90^\circ - \frac{\theta}{2}\right] \\ &= 90^\circ - \zeta_1 - \Delta\zeta + \frac{\theta}{2}. \qquad (g)\end{aligned}$$

Taking the mean of (f) and (g)

$$\alpha = \frac{\zeta_2 - \zeta_1}{2}. \tag{h}$$

In the triangle P_1P_2O

$$\beta = \zeta_1 + \Delta\zeta - \theta. \tag{i}$$

Also,
$$\beta = 180^\circ - \zeta_2 - \Delta\zeta. \tag{j}$$

* The following formulæ are those adopted by the Coast and Geodetic Survey in 1915 (see Special Publications Nos. 26 and 28).

Taking the mean of (i) and (j),

$$\beta = 90^\circ - \left(\frac{\theta}{2} + \frac{\zeta_2 - \zeta_1}{2}\right). \tag{k}$$

Substituting (h) and (k) in (e),

$$h_2 - h_1 = 2\,(R_\alpha + h_1)\sin\frac{\theta}{2}\,\frac{\sin\left(\frac{\zeta_2 - \zeta_1}{2}\right)}{\cos\left(\frac{\theta}{2} + \frac{\zeta_2 - \zeta_1}{2}\right)} \tag{l}$$

Expanding the denominator and dividing both numerator and denominator by $\cos\left(\frac{\zeta_2 - \zeta_1}{2}\right)\cos\frac{\theta}{2}$, we obtain

$$h_2 - h_1 = \frac{2\,(R_\alpha + h_1)\tan\frac{\theta}{2}\tan\left(\frac{\zeta_2 - \zeta_1}{2}\right)}{1 - \tan\frac{\theta}{2}\tan\left(\frac{\zeta_2 - \zeta_1}{2}\right)}$$

Expanding $\tan\frac{\theta}{2}$ in series (see p. 408), retaining two terms in the numerator and one term in the denominator, we have, putting $\frac{s}{R_\alpha}$ for θ,

$$h_2 - h_1 = \frac{\left(1 + \frac{h_1}{R_\alpha}\right)s\tan\left(\frac{\zeta_2 - \zeta_1}{2}\right)\left(1 + \frac{s^2}{12\,R_\alpha^2}\right)}{1 + \frac{s\tan\left(\frac{\zeta_2 - \zeta_1}{2}\right)}{2\,R_\alpha}} \tag{128}$$

$$= s\tan\left(\frac{\zeta_2 - \zeta_1}{2}\right)\cdot A\cdot B\cdot C \tag{129}$$

in which

$$A = 1 + \frac{h_1}{R_\alpha}$$

the correction for elevation of the station of known elevation,

$$B = 1 + \frac{}{2\,R_\alpha}\tan\left(\frac{\zeta_2 - \zeta_1}{2}\right)$$

the correction for the difference in elevation, and

$$C = 1 + \frac{s}{12 R_\alpha^2},$$

the correction for distance.

The logarithms of A, B, and C are given in Tables K, L, and M, for the arguments h_1, $\log\left[s \tan\left(\frac{\zeta_2 - \zeta_1}{2}\right)\right]$, and $\log s$, respectively.

174. When only one Zenith Distance is Observed.

From (g) and (h) we have

$$\alpha = \frac{\zeta_2 - \zeta_1}{2} = 90^\circ - \zeta_1 - \Delta\zeta + \frac{\theta}{2}.$$

The refraction angle is $\Delta\zeta = m\theta$, where m is the refraction coefficient, to be determined by actual observation. This coefficient is on the average nearly equal to 0.071. Substituting $m\theta$ for $\Delta\zeta$ in the preceding equation we have

$$\frac{\zeta_2 - \zeta_1}{2} = 90^\circ - \zeta_1 + (0.5 - m)\,\theta,$$

and
$$\tan\frac{\zeta_2 - \zeta_1}{2} = \tan\left[90^\circ + (0.5 - m)\frac{s}{R_\alpha} - \zeta_1\right].$$

Reducing the small term to seconds,

$$\tan\frac{\zeta_2 - \zeta_1}{2} = \tan\left[90^\circ + (0.5 - m)\frac{s}{R_\alpha \sin 1''} - \zeta_1\right],$$

$$= \tan\,[90^\circ + k - \zeta_1]. \qquad (n)$$

Substituting (n) in [129],

$$h_2 - h_1 = s \tan\,[90^\circ + k - \zeta_1]\, A \cdot B \cdot C \qquad [130]$$

in which A, B, and C have the same meaning as before except that B is given for the argument $\log\,[s \tan\,(90^\circ + k - \zeta_1)]$.

Example. Zenith Distance of Mt. Blue from Farmington, 87° 07′ 18″.8; distance, 15,519 meters; m = 0.071; instrument 2.20 meters above station mark; point sighted 4.40 meters above station mark: elevation of Farmington, 181 20 meters.

$$
\begin{array}{rl}
 & 0.5 \\
m & \underline{0.071} \\
(0.5-m) & 0.429
\end{array}
\qquad
\begin{array}{rcl}
\log & = & 9.6325 \\
\log s & = & 4.1909 \\
\text{colog } R_\alpha \sin 1'' & = & \underline{8.5092} \\
 & & 2.3326 \\
k = 215''.1 & &
\end{array}
$$

$$
\begin{array}{lcl}
\log R_\alpha & = & 6.8052 \\
\text{“} \sin 1'' & = & \underline{4.6856} \\
 & & 1.4908
\end{array}
\qquad
\begin{array}{rcr}
 & & 90^\circ\ 00'\ 00'' \\
k = 215''.1 & = & \underline{03'\ 35''.1} \\
\zeta & = & \underline{87^\circ\ 07'\ 18''.8} \\
 & + & 2^\circ\ 56'\ 16''.3
\end{array}
\qquad
\begin{array}{rcl}
\tan & = & 8.71029 \\
\log s & = & \underline{4.19086} \\
 & & 2.90115 \\
A & & 1 \\
B & & 3 \\
C & & \underline{0} \\
 & & 2.90119
\end{array}
$$

	796.51	meters
Red. to Sta.	2.20	“
Diff. Eleva	794.31	“
Elev. Farmington	181.20	“
Elev. Mt. Blue	975.51	“

TABLE K

h_1	Log A, units of fifth place of decimals	h_1.	Log A, units of fifth place of decimals	h_1.	Log A, units of fifth place of decimals.	h_1	Log A, units of fifth place of decimals.
Meters.		Meters.		Meters		Meters	
0	0	1541	11	3156	22	4770	33
73	1	1688	12	3303	23	4917	34
220	2	1835	13	3449	24	5064	35
367	3	1982	14	3596	25	5211	36
514	4	2128	15	3743	26	5357	37
661	5	2275	16	3890	27	5504	38
807	6	2422	17	4036	28	5651	39
954	7	2569	18	4183	29	5798	40
1101	8	2715	19	4330	30	5945	41
1248	9	2862	20	4477	31	6091	
1394	10	3009	21	4624	32		
1541		3156		4770			

* In these tables log R_α is taken as 6.80444, the mean radius in latitude 40° on the Clarke Spheroid of 1866.

Table K gives the values of log A, the correction factor for the elevation of the known station, by showing the limiting values of the elevation h_1, between which log A may be taken as 0, 1, 2, 3, etc., units of the fifth place of decimals. Log A is positive, except in the very rare case where h_1 corresponds to a point below mean sea-level.

TABLE L

Log s tan $\frac{1}{2}$ $(\zeta_2 - \zeta_1)$ or log s tan $(90° + k - \zeta_1)$ · (s in meters.)	Log B, units of fifth place of decimals.	Log s tan $\frac{1}{2}$ $(\zeta_2 - \zeta_1)$ or log s tan $(90° + k - \zeta_1)$ · (s in meters.)	Log B units of fifth place of decimals	Log s tan $\frac{1}{2}$ $(\zeta_2 - \zeta_1)$ or log s tan $(90° + k - \zeta_1)$ · (s in meters.)	Log B units of fifth place of decimals
	0				
2.167		3 397		3 685	
	1		9		17
2.644		3 445		3 711	
	2		10		18
2 866		3 489		3 735	
	3		11		19
3 011		3 528		3 758	
	4		12		10
3.121		3 565		3 779	
	5		13		21
3 208		3 598		3 800	
	6		14		22
3.281		3 629		3 820	
	7		15		23
3.343		3.658		3 839	
	8		16		24
3.397		3.685		3 857	

Table L gives the values of log B, the correction factor for approximate difference of elevation by showing the limiting values of log $[s \tan \frac{1}{2} (\zeta_2 - \zeta_1)]$ or log $s \tan (90° + k - \zeta_1)]$ between which log B may be taken as 0, 1, 2, 3, etc., units of the fifth place of decimals. Log B has the same sign as the angle $\frac{1}{2} (\zeta_2 - \zeta_1)$ or $90° + k - \zeta_1$; for example, if log $[s \tan \frac{1}{2} (\zeta_2 - \zeta_1)]$ lies between 3.565 and 3.598 and $\frac{1}{2} (\zeta_2 - \zeta_1)$ is positive, log B = +0.00013, but if $\frac{1}{2} (\zeta_2 - \zeta_1)$ is negative then log B = −0.00013, *i.e.*, 9.99987 − 10, the former way of writing being usually more convenient in practice.

TABLE M

Log s (s in meters).	Log C, units of fifth place of decimals.	Log s (s in meters)	Log C, units of fifth place of decimals.
0.000		5.297	
	0		4
4.875		5.352	
	1		5
5.113		5.395	
	2		6
5.224		5.432	
	3		7
5.297		5.463	

Table M gives the value of log C, the correction factor for distance between stations, by showing the limiting values of log s between which log C may be taken as 0, 1, 2, 3, etc., units of the fifth place of decimals. Log C is always positive.

Determining the Coefficient, m.

In deriving formula [129] it was assumed that since the rays travel over the same path in opposite directions simultaneously the values of $\Delta\zeta$ may be considered equal to each other and hence do not appear in the final equation. Whenever the difference in elevation has been calculated from reciprocal observations it then becomes possible to find what value of m will satisfy the equation for this particular line. If in [130] we substitute the known difference in elevation $h_2 - h_1$ and, regarding the k term as unknown, solve for the value of m, we then have a value consistent with the observed angles.

Suppose that the difference in elevation of two stations is 1046.9 meters, the distance is 23931.6 meters, and the angles (from the zenith) are $87° 35' 01''.1$ for the lower station and $92° 35' 34''.2$ for the upper station. The elevation of the lower station is 108.87.

Solving equation [130] we have,

$$
\begin{aligned}
\log 1046.9 &= 3.01991 \\
A &= 1 \\
B &= 4 \\
C &= \underline{0} \\
 & 3.01986 \\
\log s &= \underline{4.37897} \\
\log \tan (90^\circ + k - \zeta_1) &= 8.64089 \\
(90^\circ + k - \zeta_1) &= 2^\circ\ 30'\ 16''.5 \\
90^\circ - \zeta_1 &= \underline{2\ \ 24\ \ 58\ .9} \\
k &= 0^\circ\ 05'\ 17''.6 \\
&= 317''.6 \\
\log k &= 2.50188 \\
\log \frac{s}{R_\alpha \sin 1''} &= \underline{2.88870} \\
\log (0.5 - m) &= 9.61318 \\
0.5 - m &= 0.4104 \\
m &= 0.0896
\end{aligned}
$$

The value of m as found from a large number of observations is given in the reports of the Coast and Geodetic Survey as follows:

Lines crossing the sea	0.078
Between high stations	0.071
In the interior of the country	0.065

In Clarke's *Geodesy* are found,

For rays	crossing the sea	.0809
" " not	" " "	.0750

These values are but averages. The actual values vary considerably with atmospheric conditions. Some lines near the surface can be sighted on certain days, while on other days the signals are entirely invisible. At Mt. Diablo, Calif., the following variations were observed: 3^h A.M., .0893; 9^h A.M., .0812; 2^h P.M., .0640; 9^h P.M., .0827.

PROBLEMS

Problem 1. Calculate the orthometric correction for a line of levels extending 2° northward from a point in latitude 45° N at an elevation of 1000 meters.

Problem 2. Compute the correction for reducing to sea-level a latitude observed at an elevation of one mile in latitude 45° N.

Problem 3. Vertical angle from S to B, $+ 2^\circ\ 24'\ 58''.94$. Vertical angle from B to S, $-2^\circ\ 35'\ 34''.20$. Elevation of S = 108.87 meters; distance, 23,931.6 meters. Compute the elevation of B.

CHAPTER XI

MAP PROJECTIONS

175. Map Projections.

Whenever we attempt to represent a spherical or a spheroidal surface on a plane some distortion necessarily results, no matter how small may be the area in question. The problem to be solved in constructing topographic or hydrographic maps is to find a method which will minimize this distortion under the existing conditions. The number of projections which have been devised is very great; for the description and the mathematical discussion of the properties of these projections the reader is referred to such works as Thomas Craig's *Treatise on Projections, United States Coast and Geodetic Survey*, 1882; *The Coast and Geodetic Survey Report*, 1880; C. L. H. Max Jurisch, *Map Projections*, Cape Town, 1890; G. James Morrison, *Maps, Their Uses and Construction*, London, 1902; A. R. Hinks, *Map Projections*, Cambridge, 1912; and Special Publ. 47, 49, 52, and 130, U. S. Coast and Geodetic Survey.

In this chapter we shall consider only those projections which are used for such maps and charts as are of importance in geodetic surveys and in navigation.

176. Simple Conic Projection.

In this projection the map is conceived to be drawn on the surface of a right circular cone which is tangent to the sphere or the spheroid along a single parallel of latitude, usually the middle latitude. The apex of the cone lies in the prolongation of the axis of the spheroid. From Fig. 138 it is evident that the distance TA from the apex to the parallel through A is equal to $N \cot \phi$. If the cone is developed on a plane surface we shall have a sector whose center is T and whose radius is $N \cot \phi$.

(Fig. 139.) All other parallels of latitude on the map will be circles drawn about the same center T, and all meridians will be represented by straight lines passing through T. The spacing between the parallels of latitude is obtained by laying off distances along the central meridian which are proportional to the distances between the same parallels on the spheroid. The position of the meridians is found by subdividing the middle

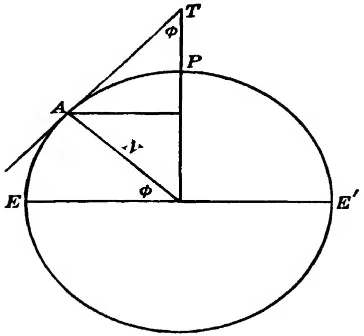


FIG. 138.

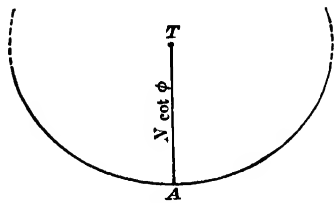


FIG. 139.

parallel into spaces which are proportional to the lengths of the arcs of the same parallel on the spheroid. Straight lines are then drawn from the center T through these points of sub-division. Any meridian or any parallel may be assumed for the central meridian and middle parallel of the map. It is evident from the above that this is not a true projection, that is, the points are not those that would be obtained by projecting radially from the center of the sphere onto the cone. If the scale of the map is such that the position of the center T cannot be represented on the paper, the curves may be laid off by plotting certain points by means of their rectangular coördinates as described later under the polyconic projection. Tables X, XV, XVI and XVII may be used for this purpose.

It is evident that the meridians and parallels of a conic pro-

jection intersect at right angles in all parts of the map, as they do on the sphere. The scale of the map is not correct, however, except along the middle parallel. For a map having a great extension in the longitude and but little in the latitude, the conic projection is fairly accurate. Figure 140 shows a completed conic projection covering the area of the United States.

177. Bonne's Projection.

This projection is a modification of the simple conic and meets the objection that the scale of the latter becomes inaccurate as the distance from the middle parallel increases. The parallels of latitude are concentric circles as before, but *each* parallel is sub-divided into spaces which are proportional to the corresponding spaces on that parallel on the spheroid. The central meridian and all parallels are therefore correctly sub-divided. The meridians are obtained by joining the points of sub-division on the parallels. The meridians in this projection are all curved, except the central one, and they intersect the parallels nearly, but not quite, at right angles (Fig. 141). The distortion in this projection is very small, and for small areas it is practically a perfect projection. It has been much used in Europe.

178. The Polyconic Projection.

The idea of using several cones, or the polyconic projection, is due to Mr. F. R. Hassler, the first superintendent of the Coast Survey. Each parallel of latitude shown on the map is developed on a cone tangent along that parallel. The radius (TA) for any parallel (latitude ϕ) is N cot ϕ; and the angle between two elements of the cone when developed is approximately $\theta = (d\lambda) \sin \phi$, as will be evident from Fig. 142. See also Equa. [68], p. 218.

In constructing the map the degrees of latitude are laid off along the central meridian, the spacing corresponding to the distances on the spheroid. (Table X.) The points where the meridians intersect the parallels are plotted by means of their rectangular coördinates, the coördinate axes being in each case the central meridian and a line at right angles to it drawn through

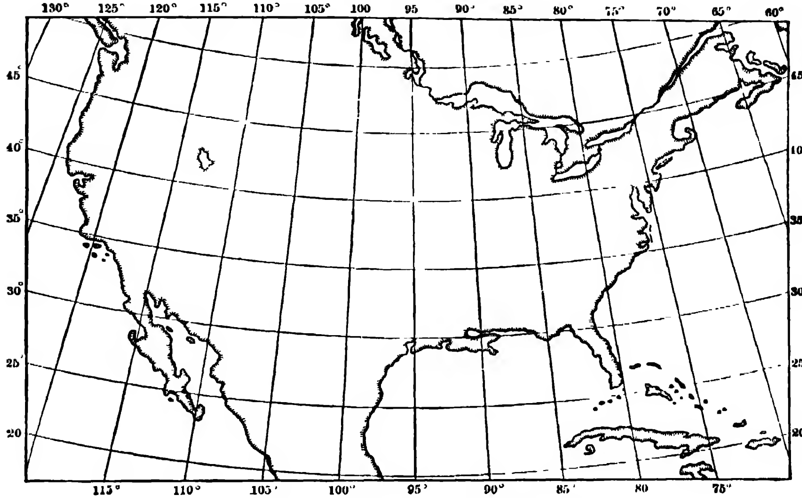


Fig. 140. Simple Conic Projection.

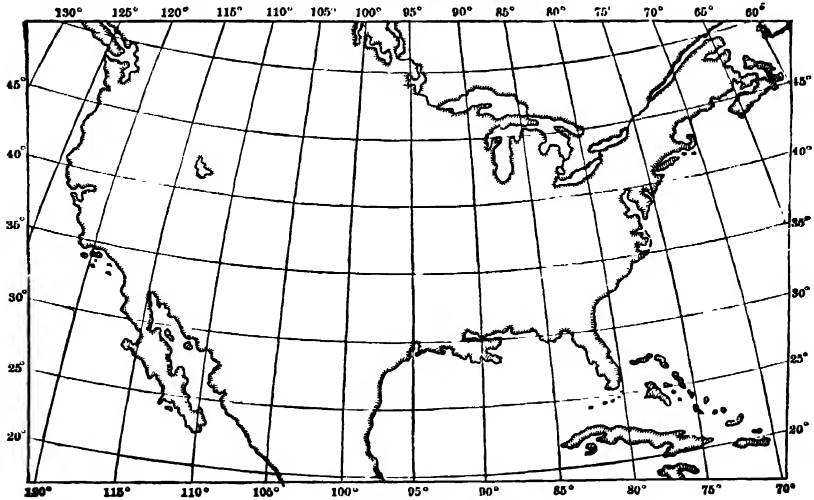


Fig. 141. Bonne's Projection.

the latitude in question. The coördinates themselves are found as follows: In Fig. 143, let A be the intersection of some meridian and parallel which are to be drawn on the map. Then the radius $TA = N \cot \phi$ may be computed from the known latitude of A, and the angle θ may be computed from the known

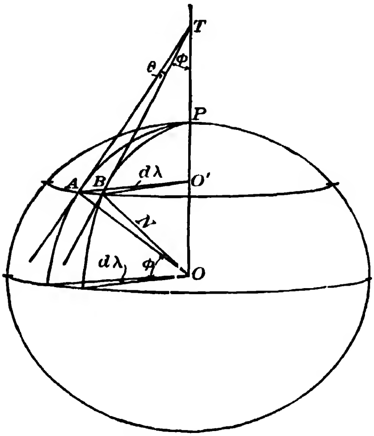


FIG. 142.

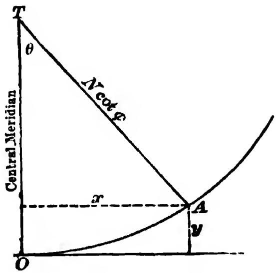


FIG. 143.

difference in longitude between O and A by the equation $\theta = (d\lambda) \sin \phi$. Then for x and y we have

$$x = TA \sin \theta = N \cot \phi \sin (d\lambda \sin \phi) \qquad [131]$$

and

$$\begin{aligned} y &= TA \operatorname{vers} \theta = \frac{x}{\sin \theta} \operatorname{vers} \theta \\ &= x \tan \frac{\theta}{2} \\ &= x \tan \tfrac{1}{2} (d\lambda \sin \phi) \qquad [132] \end{aligned}$$

Values of these numbers will be found in Tables XVI and XVII which are taken from a large general table in Coast and Geodetic Survey Special Publ. No. 5.

It is evident that the parallels and meridians do not intersect at right angles except at the central meridian. The meridian

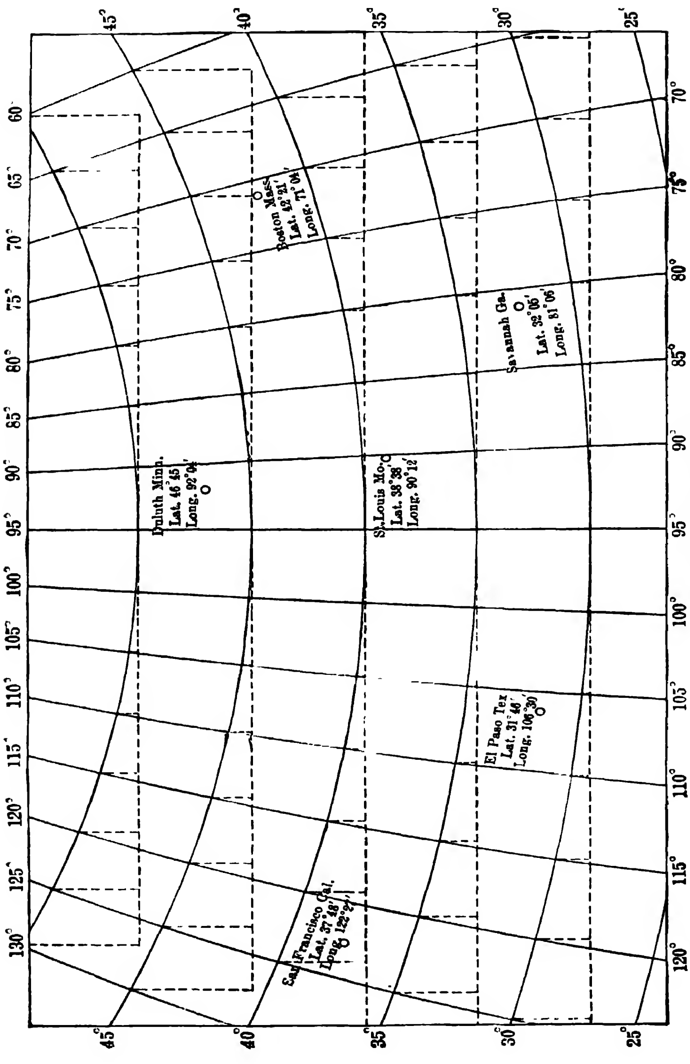
and parallels are both curved, as in Bonne's projection, but since the lower parallels are flatter there is a separation of the parallels which becomes more marked toward the east and west margins of the map. For this reason this map becomes less and less accurate as the longitude is extended. In mapping areas which extend principally north and south, it is superior to other projections. It is in general use in the United States for Government maps. Figure 144 shows a polyconic projection covering the area of the United States.

Another advantage of the polyconic projection is that it is adapted for use in any part of the globe. If a map is to be made in any region a central meridian is selected and the limits of latitude determined. Then the same tables of coördinates may be used for this map without any special computations. In any form of projection in which the standard parallels are fixed, such as the Lambert or the Albers projections, this cannot be done; the map must still refer to the same standard parallels, which may be far from the area being mapped.

There is one disadvantage in the Polyconic and the Bonne's projections, namely, that if two maps of adjoining areas are to be placed side by side they cannot be placed exactly in contact because the limiting (common) meridian curves in opposite directions on the two maps. In the simple conic and in the Lambert projection, to be described in the next article, the meridians are straight and this difficulty does not exist.

179. Lambert's Conformal Projection.

The Lambert projection having two standard parallels was invented about the middle of the eighteenth century, but has recently been brought into prominence through its use in the French battle maps. The fundamental notion is that of a cone tangent along the middle parallel of the map, the radius of this parallel (on the map) being $N \cot \phi$, and the angle between the central meridian and any other meridian being $(d\lambda) \sin \phi$. This would give a map in which one parallel, and only one, is correctly divided. We may, however, modify the projection



Longitude West from Greenwich.

FIG. 144. Polyconic Projection.

so as to have two standard (correct) parallels. This is done by reducing the scale (multiplying by a constant) and is practically equivalent to employing a cone which cuts the spheroid in the two standard parallels.

The other parallels are so spaced that the scale of the map is the same for all azimuths at any one place, that is, the scale along a meridian is the same as the scale in an east and west plane. A projection having this property is said to be "conformal." It may be proved that this condition is true if the spacing between parallels is $\beta + \frac{\beta^3}{6\rho_0^2}$, where β is the arc of the meridian between parallels on the original tangent cone measured from the parallel of contact, and ρ_0 is the mean radius of curvature of the spheroid at a point on this tangent parallel.

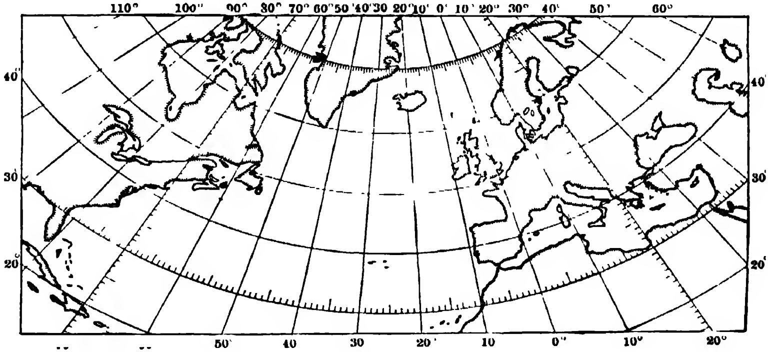


FIG. 145. Lambert Projection.

Since the projection is conformal, all lines on the map cut each other at the same angles as do the corresponding lines on the spheroid. There is a tendency, therefore, for small figures to have the same shape on the map that they have on the earth's surface. The scale of this map is correct on the two standard parallels. Between these two parallels the scale is a little too small and outside these parallels the scale is too large. The

error is not serious, however, if the standard parallels are chosen, as is usual, one sixth and five sixths the length of the meridian arc to be shown. Figure 145 shows a Lambert projection.

This projection may be extended indefinitely in an east and west direction without error. The error becomes greater and greater as the map is extended to the north and south. In this respect it is just the contrary of the polyconic projection. As compared with the polyconic projection for an area like that of the United States proper, the Lambert will give long diagonal distances (Maine to Arizona, for example) with a much smaller scale error than the polyconic. For a complete description of this projection, together with tables for projecting maps, see United States Coast Survey Special Publications 47, 49 and 52.

179a. Albers Equal Area Projection.

This is a conical projection in which the meridians are shown by straight lines and the parallels by concentric circles. It is also an "equal area" projection, that is, areas on the map are proportional to the areas on the globe. The scale along two standard parallels is exact, as in the Lambert. This projection not only has a correct scale along the two standard parallels, but it has a minimum error along the borders and has no scale errors along two curves which lie close to the diagonals of the map. On the whole it has a smaller scale error than any other projection that has been used. This projection is well adapted for constructing a large map of the whole United States. (See U. S. Coast and Geodetic Special Publ. No. 130.)

180. The Gnomonic Projection.

In the gnomonic, or central, projection the projecting point is at the center of the sphere and the plane of the map is tangent to the sphere at some selected point. Every plane through the center cuts the sphere in a great circle and cuts the map in a straight line; hence every great circle is represented by a straight line and every straight line on the map must represent a great circle.

Figure 146 shows the Atlantic Ocean projected on a plane tangent at $\phi = 30°$ N and $\lambda = 30°$ W.

The meridians and the equator are of course represented by straight lines. The parallels of latitude are conic sections, in this case hyperbolas. The parallels are best constructed by employing the equations of the curves and plotting points by means of coördinates.

If a gnomonic chart is constructed on a plane tangent at the earth's pole the construction is quite simple. The angles between

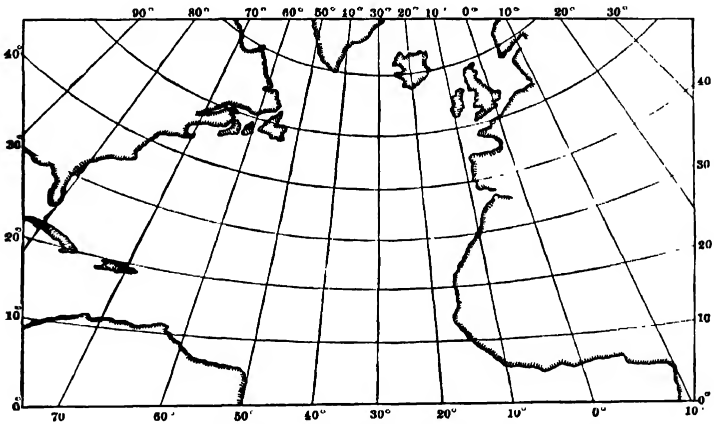


FIG. 146. Gnomonic Projection or Great-circle Chart.

meridians are equal to the actual longitude differences, and the radius of any parallel of latitude is given by $R \cot \phi$, where R is the radius of the sphere.

The gnomonic projection is used almost exclusively for determining the positions of great circles for the purposes of navigation. By joining any two places by a straight line the great-circle (or shortest) track is at once shown. The latitudes and longitudes of any number of points on this track may be read off the chart and, if desired, may be transferred to any other

chart and the curve sketched in. The point where the great circle approaches most nearly to the pole is found at once by drawing from the pole a line perpendicular to the track. The foot of this perpendicular is the vertex, or point of highest latitude.

181. Cylindrical Projection.

If a cylinder is circumscribed about a sphere so as to be tangent along the equator, and if points be projected onto the cylinder by straight lines from the center, the cylinder, when developed will give a map in which the meridians and parallels are all straight lines cutting each other at right angles, the relative distances between points being approximately correct near the equator but distorted in high latitudes. The meridians will all be parallel to each other. The parallels of latitude will be parallel to each other and will be spaced wider and wider apart as the latitude increases. Evidently the scale of the map is different for different latitudes. It is also true that at any point the scale along a meridian is not the same as the scale along a parallel. Such a projection is of no practical value, but its description aids in understanding the Mercator projection which is described in the next article.

182. Mercator's Projection.

A modification of the above projection, known as Mercator's, consists in so spacing the parallels of latitude that the relation between increments of latitude and longitude on the chart is the same as the relation between increments of latitude and longitude at the corresponding point on the earth's surface, or approximately, $1'$ lat. on chart: $1'$ long. on chart $=$ $1'$ lat. on spheroid: $1'$ long. on spheroid. If this relation is preserved, it will be found that any line of constant bearing (called the ***loxodrome*** or ***rhumb line***) will be represented by a straight line on the chart.

In Fig. 147 let AB on the earth's surface be represented by $A'B'$ on the chart (actual size). In order that the two lines may have the same bearing it is necessary that

$$\frac{dy}{dx} = \frac{AC}{CB} = \frac{R_m d\phi}{R_p d\lambda}$$

or
$$dy = \frac{dx}{R_p d\lambda} \cdot R_m d\phi. \tag{a}$$

In other words, since the longitude has been expanded $\left(\text{in the ratio } \frac{a}{R_p}\right)$ by the method of constructing the chart, it is necessary to expand the latitudes in the same ratio in order to preserve

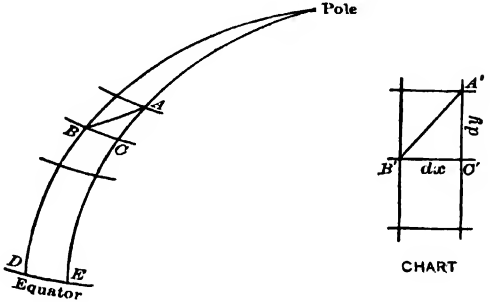


FIG. 147.

the scale and give AB the same bearing. Now since dx is represented as large as the corresponding arc on the equator, we have

$$\frac{dx}{R_p d\lambda} = \frac{a\, d\lambda}{R_p d\lambda} = \frac{a}{R_p}.$$

Substituting in (a), we obtain

$$dy = \frac{a}{R_p} \cdot R_m d\phi$$

or, since $R_p = N \cos \phi$

$$dy = \frac{R_m}{N \cos \phi} \quad a\, d\phi$$

$$\frac{a\,(1 - e^2)}{\cos \phi\,(1 - e^2 \sin^2 \phi)} \cdot d\phi$$

$$\therefore \quad y = a \int_0^{\phi} \frac{(1 - e^2)}{\cos \phi (1 - e^2 \sin^2 \phi)} \cdot d\phi.$$

Multiplying e^2 by $\sin^2\phi + \cos^2\phi$, the integral may be separated into two, giving, after multiplying numerator and denominator by $\cos\phi$,

$$y = a\int_0^\phi \frac{\cos\phi\, d\phi}{\cos^2\phi} \cdot - ae\int_0^\phi \frac{e\cos\phi\, d\phi}{1 - e^2\sin^2\phi}$$

$$= \frac{a}{M}\left[\tfrac{1}{2}\log\frac{1+\sin\phi}{1-\sin\phi} - \tfrac{1}{2}e\log\frac{1+e\sin\phi}{1-e\sin\phi}\right]_0^\phi$$

where $M = 0.4342945$, the modulus of the common logarithms.

Employing the formulæ,

$$\log\frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \cdots\right)$$

and

$$\frac{1+\sin x}{1-\sin x} = \tan^2\left(45^\circ + \frac{x}{2}\right)$$

the equation may be expressed

$$y = \frac{a}{M}\left[\log\tan\left(45^\circ + \frac{\phi}{2}\right)\right]_0^\phi - ae\left[e\sin\phi + \frac{(e\sin\phi)^3}{3} + \cdots\right]_0^\phi \quad [133]$$

in which y is the same linear units as a.

In order to express y in nautical miles or minutes of arc on the equator* it is necessary to multiply by $\dfrac{60 \times 180}{a\pi}$, giving,

$$y = 7915.705\log\tan\left(45^\circ + \frac{\phi}{2}\right) - 3437.7\left(e^2\sin\phi + \frac{e^4\sin^3\phi}{3}\right), \quad [134]$$

or

$$y = 7915.705\log\tan\left(45^\circ + \frac{\phi}{2}\right) - 22'.945\sin\phi - 0.051\sin^3\phi. \quad [135]$$

Also $$x = 60 \times \lambda^\circ, \quad [136]$$

the unit being the nautical mile. Values of y, called *meridional parts*, will be found in works on navigation. (Bowditch, Table 3.)

* The Nautical Mile contains 6080.20 ft., (Clarke Spheroid and U. S. legal meter); this is not identical with the number of feet in one minute of arc on the earth's equator. For a discussion of this matter, see Appendix 12, Coast Survey Report for 1881.

This chart is much used by navigators because it possesses the property that the bearing of any point B from a point A as measured on the chart is the same as that bearing on which a vessel must sail continuously to go from A to B. The track

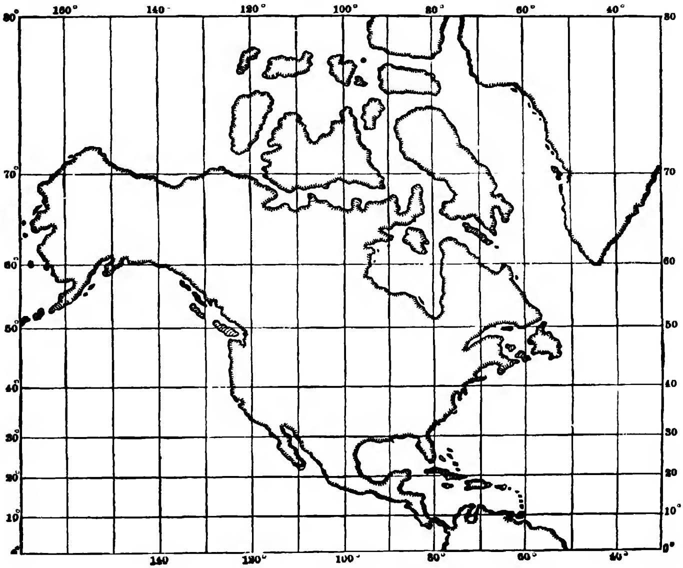


FIG. 148. Mercator Chart.

cuts all meridians on the globe at the same angle, just as a straight line on the chart cuts all meridians at the same angle. This track is not the shortest one between A and B, but for ordinary distances the length differs but little from that of the great-circle track. In following a great-circle track the navigator may transfer to the Mercator chart a few points on the

great-circle obtained from his great-circle chart, by means of their latitudes and longitudes, and then sail on the rhumb lines between consecutive plotted points. Figure 148 shows a Mercator chart.

183. Rectangular Spherical Coördinates.

A system of rectangular spherical coördinates, used in Europe, consists in referring all points to two great circles through some selected origin, one of them being the meridian, the other the prime vertical. Within small areas these coördinates are practically the same as rectangular plane coördinates. When the area is so great that the effect of curvature becomes appreciable, small corrections are introduced, so that the form of the plane coördinates is retained without loss of accuracy. Such a system is very convenient when connecting detail surveys with the triangulation, particularly for local surveyors who may not be familiar with geodetic methods of calculating latitudes and longitudes. The method is not well adapted to mapping very large areas. (See Crandall's *Geodesy*, p. 187.)

CHAPTER XII

APPLICATION OF METHOD OF LEAST SQUARES TO THE ADJUSTMENT OF TRIANGULATION

184. Errors of Observation.

Whenever an observer attempts to determine the values of any unknown quantities, he at once discovers a limit to the precision with which he can make a single measurement. In order to secure greater precision in his final result than can be obtained by a single measurement, he resorts to the expedient of making additional measurements, either under the same conditions or under different conditions. Under these circumstances it will be observed that the results are discordant and that the same numerical result almost never occurs twice.* The question at once arises, then, What are the best values of the unknown quantities which it is possible to obtain from these measurements?

The method of least squares has for its main objects (1) the determination of the best values which it is possible to obtain from a given set of measurements, and (2) the determination of the degree of dependence which can be placed upon these values, or, in other words, the relative worth of different determinations; (3) it also enables us to trace to their sources the various errors affecting the measurements and consequently to increase the accuracy of the result by a proper modification of the methods and instruments used. The method is founded upon the mathematical theory of probability, and upon the

* This is only true, however, when the observer is taking each reading with the utmost possible refinement. If, for example, angles are read only to the nearest degree, the result will always be the same no matter how many times the measurement may be repeated; but if read to seconds and fractions, they will in general all be different.

assumption that those values of the unknowns which are rendered most probable are the best that can be obtained from the measurements.

185. Probability.

If an event can happen in a ways and fail in b ways, and all of these ways are equally likely to occur, the probability that the event will happen in any one trial is expressed by the fraction $\frac{a}{a+b}$, and the probability that it will fail is expressed by $\frac{b}{a+b}$. Since it must either happen or fail, the sum of the two probabilities represents a certainty. This sum is $\frac{a}{a+b} + \frac{b}{a+b} = 1$. Therefore the probability of the happening of an event is represented by some number lying between 0 and 1, the larger the fraction the greater the probability of its happening. For example, a die may fall so that any one of its six faces is uppermost, and all of these six possibilities are equally likely to occur; the probability of any one of its faces being up is $\frac{1}{6}$.

186. Compound Events.

If a certain event can happen in a ways and fail in b ways, and if a second, independent, event can happen in a' ways and fail in b' ways, and all are equally likely to occur, then the total number of ways in which the events can take place together is $(a + b)(a' + b')$. The number of ways in which both can happen is aa' and the probability of its happening is $\frac{aa'}{(a+b)(a'+b')}$. For example, the probability of double six being thrown with a pair of dice is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$. It is evident that the probability of the simultaneous occurrence of two events is the product of the probabilities of the occurrence of the component events. In a similar way it may be shown that the probability of the simultaneous occurrence of any number of independent events is the product of their separate probabilities; that is, if P_1, P_2, P_3 . . . are the probabilities of the occurrence of any number of independent events, the probability of their simultaneous

occurrence is

$$P = P_1 \times P_2 \times P_3 \ldots, \qquad [137]$$

187. Errors of Measurement — Classes of Errors.

Every measurement of a quantity is subject to error, of which the following kinds may be distinguished.

1. Constant Errors.
2. Systematic Errors.
3. Accidental Errors.

188. Constant Errors.

A constant error has the same effect upon all observations in the same series of measurements. For instance, if a steel tape is 0.01 ft. too long, this error affects every 100 ft. measurement in just the same way.

189. Systematic Errors.

A systematic error is one of which the algebraic sign and the magnitude bear a fixed relation to some condition. For example, if the measurements with the tape are made at different temperatures, the error resulting from this variation of temperature is systematic and may be computed if the temperatures and the coefficient of expansion are known.

190. Accidental Errors.

Accidental errors are not constant from observation to observation; they are just as likely to be positive as negative; in general they follow the exponential law of error, as will be explained later (Art. 197). The error of placing a mark opposite to the end graduation of the tape is of this class.

191. Comparison of Errors.

There is in reality no fixed boundary between the accidental and the systematic errors. Every accidental error has some cause, and if the cause were perfectly understood and the amount and sign could be determined, it would cease to be an accidental error, but would be classed as systematic. On the other hand, errors which are either constant or systematic may be brought into the accidental class, or at least made partially to obey the

law of accidental error, by so varying the conditions, instruments, etc., that the sign of the error is frequently reversed. If a tape has 0.01 ft. uncertainty in length, this produces a constant error in the result of a measurement. If, however, we use several different tapes, each with an uncertainty of 0.01 ft. this error may be positive or negative in any one case. In the long run these different errors tend to compensate each other like accidental errors.

In the class of systematic errors would be placed such errors as those due to changes in temperature, light, and moisture, or change in the adjustments of instruments. These errors may be computed and allowed for as soon as we know the law governing their action, or they may be partially eliminated by varying conditions under which the measurements are made.

Under the constant class comes the observer's error, which tends to become constant with increased experience in observing. This error may be allowed for as soon as its magnitude and sign have been determined, or it may be eliminated by the method of observation. Certain errors in the instrument may have a constant effect on the result; these may be dealt with in the same manner as the personal error. It should be noticed that after the constant error or the systematic error has been eliminated, there still remains a small error due to the fact that the magnitude of the constant error itself was not perfectly determined or that its elimination was imperfect. This remaining error must be regarded as an error of the accidental class, since its magnitude is unknown and it is just as likely to be positive as negative.

Under accidental errors are included all those which are supposed to be small and just as likely to be positive as negative. They are due to numerous unknown causes, each error being in reality the algebraic sum of many smaller errors. Under this class may be noted errors in pointing with a telescope, errors in reading scales and estimating fractions of scale divisions, and

undetected variations in all of the conditions governing systematic errors.

192. Mistakes.

These are not errors, but they must be considered in connection with the discussion of accuracy of observations. They include such cases as reading one figure for another, as a 6 for a 0, or reading a scale in the wrong direction, as reading 46° for 34°.

193. Adjustment of Observations.

When the number of measurements is just sufficient to determine the quantities desired, then there is but one possible solution, and the results must be accepted as the true values. When additional measurements are made for the purpose of increasing the accuracy of the results, this gives rise to discrepancies among the different measurements of the same quantities, since each is subject to errors. The method of least squares enables us to compute those values which are rendered most probable by the existence of the observations and in view of the discrepancies noted; it cannot, however, tell us anything about the existence of constant errors, unless new observations made under different conditions reveal new discrepancies. For example, if a pendulum is swung and certain small variations in the last decimal place of the period are noticed, these may be regarded as due to small errors in the running of the chronometer and to accidental errors of observing; but if the pendulum case be mounted on a support whose flexibility is very much greater than that of the first, and larger variations are now observed, it becomes apparent that an error of the systematic class is affecting all our observations, though it does not appear at all in the first observations, because all the measurements were affected alike. An investigation of the law governing this error, and the determination of its magnitude and sign, enable us to correct the result for such part of the error as we are able to determine. There remains in the result, however, an accidental error, namely, the error in the measurement of the flexure correction.

194. Arithmetical Mean.

The formulæ employed in adjusting observations are usually made to depend upon the axiom that if a number of observations be made directly upon the same quantity, all made under the same conditions and with the same care, the most probable value of the quantity sought is the arithmetical mean of all the separate results; that is, if the results of the observations are $M_1, M_2, M_3, \ldots M_n$, the most probable value of the quantity, M_0, is given by

$$M_0 = \frac{M_1 + M_2 + \cdots M_n}{n} = \frac{\sum M}{n}. \qquad [138]$$

It is to be carefully noted that this is not the true value, M, but simply the most probable value under the circumstances; if additional measurements be made, M_0 changes correspondingly in value, because we know more about its real value than we did at first.

195. Errors and Residuals.

It now becomes necessary to distinguish between *errors* and *residuals*. The *error* is the difference between any measured value and the true value. Its magnitude can never be known, because the true value can never be known. The *residual* is the difference between a measured value and the most probable value. This is a quantity which may be computed for any set of observations. In a set of very accurate observations which are free from constant and systematic errors the residual is a close approximation to the true error. It may be shown that for the case of direct observations the algebraic sum of the residuals is zero; that is, if we compute $v_1 = M_1 - M_0$, $v_2 = M_2 - M_0$, etc., then $\sum v = 0$, where $v_1, v_2 \ldots$ are the residuals.

196. Weights.

In case the measurements are of different degrees of reliability, they are given different *weights*. The weight of an observation may be regarded as the number of times the observation is repeated and the same numerical result obtained.

It expresses the relative worth of different measured values. Weights are purely relative and may be computed on any base desired. To say that two measurements have weights 2 and 1 respectively, is the same as saying that they have weights $\frac{1}{2}$ and $\frac{1}{4}$. From the above definition it is apparent that the *weighted mean* is expressed by

$$M_0 = \frac{p_1M_1 + p_2M_2 + \cdots}{\sum p} = \frac{\sum pM}{\sum p}; \qquad [139]$$

that is, the weighted mean is found by multiplying each observation by its weight, adding the results, and dividing by the sum of the weights.

Multiplying an observation (M_1) by its weight (p_1) is the same as taking p_1 observations each equal in value to M_1.

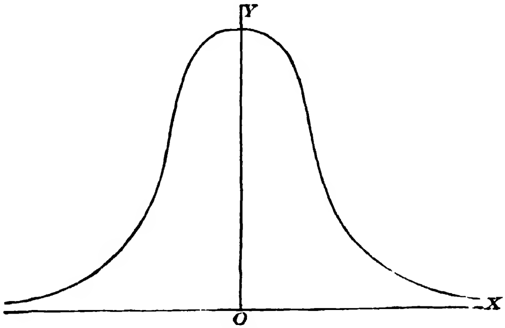


FIG. 149.

197. Distribution of Accidental Errors.

An inspection of the results of a large number of measurements will show that

(1) + and − errors are equally numerous.

(2) Small errors are much more numerous than large ones.

(3) Very large errors seldom occur.

The curve which expresses the law of variation of such errors will be of the form shown in Fig. 149. In accordance with (1)

the curve is symmetrical; in accordance with (2) its maximum is at the axis of Y; from (3) it is evident that the curve cuts the axis of X at some distance from O.

The manner in which observations are affected by accidental errors is shown by the "shot apparatus" shown in Fig. 150. A

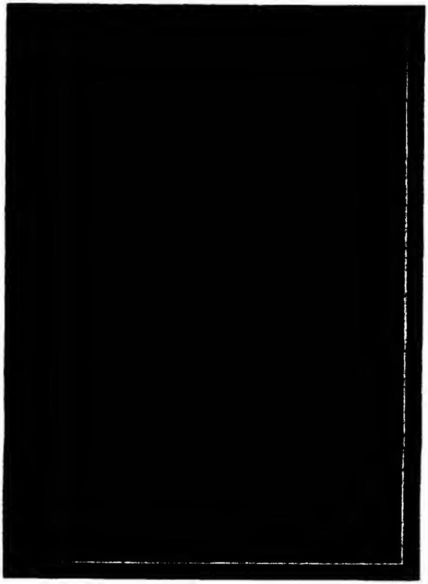


FIG. 150. "Shot Apparatus."

large number of small shot, representing observations, are allowed to drop through an opening in the middle of the case. If there were no obstructions the shot would fall directly into the central (vertical) compartment. Between the opening and the vertical compartments a number of pegs are interposed, each representing a source of error or deflection of the shot from its natural course. The shot are therefore diverted somewhat from a straight course and arrange themselves in the

different columns in the manner shown. The curve joining the tops of the columns is seen to resemble closely the "curve of error."

In order to obtain a formula expressing the law of error we suppose the curve asymptotic to the axis of X, and write the equation of the curve in the general form

$$y = f(x), \tag{140}$$

where x represents the magnitude of an error and y the frequency with which this error occurs on a large number of measurements; f represents some unknown function of x. It is necessary to assume that the number of observations is very large; otherwise the supposed balancing of $+$ and $-$ errors will be imperfect. The true error x can never be known, but the distribution of the residuals about the most probable value will evidently follow the same general law, so we may write also

$$y = f(v) \tag{141}$$

as the law to which the residuals must conform. This equation also expresses the probability of the occurrence of a residual v.

If we let the total area between the curve and the axis of X be represented by unity, then the probability that a certain residual will fall between the limits v and $v + dv$ will be represented by the area included between the curve, the X axis, and the two ordinates at v and $v + dv$, since in the long run the number in a given column will be proportional to the probability expressed by the ordinate at that point, that is,

$$y\,dv = f(v)\,dv. \tag{142}$$

If we suppose n observations of equal weight, giving the results $M_1, M_2, \ldots M_n$, to be made on any functions of the unknowns $z_1, z_2, \ldots, z_q$, giving the residuals $v_1, v_2, \ldots, v_n$, then the probability of the occurrence of these residuals is $f(v_1)\,dv$, $f(v_2)\,dv \ldots f(v_n)\,dv$. The probability of the simultaneous occurrence of these residuals is the product of the separate

probabilities, that is,

$$P = f(v_1)\,dv \times f(v_2)\,dv \times \ldots f(v_n)\,dv \qquad [143]$$

or, taking logs of both members of the equation,

$$\log P = \log f(v_1) + \log f(v_2) + \ldots \log f(v_n) + n \times \log dv.$$

The results desired for z_1, z_2, etc., are those for which the probability of the occurrence of v_1, v_2, . . . is a maximum. Therefore P must be a maximum. To find the conditions for this maximum, differentiate $\log P$ with respect to each variable, z_1, z_2, . . . , and place the results equal to zero. This gives

$$\frac{\partial \log P}{\partial z_1} = \frac{1}{f(v_1)} \cdot \frac{\partial f(v_1)}{\partial z_1} + \cdots \frac{1}{f(v_n)} \cdot \frac{\partial f(v_n)}{\partial z_1} = 0.$$

$$\frac{\partial \log P}{\partial z_2} = \frac{1}{f(v_1)} \cdot \frac{\partial f(v_1)}{\partial z_2} + \cdots \frac{1}{f(v_n)} \cdot \frac{\partial f(v_n)}{\partial z_2} = 0. \qquad [144]$$

(q equations)

But we observe that

$$\frac{\partial f(v)}{\partial z} = f'(v)\,\frac{\partial v}{\partial z}, \qquad [145]$$

in which f' represents some new function of v.

For brevity place

$$\frac{f'(v)}{f(v)} = F(v). \qquad [146]$$

Then Equa. [144] become

$$F(v_1)\,\frac{\partial v_1}{\partial z_1} + F(v_2)\,\frac{\partial v_2}{\partial z_1} + \cdots F(v_n)\,\frac{\partial v_n}{\partial z_1} = 0.$$

$$F(v_1)\,\frac{\partial v_1}{\partial z_2} + F(v_2)\,\frac{\partial v_2}{\partial z_2} + \cdots F(v_n)\,\frac{\partial v_n}{\partial z_2} = 0. \qquad [147]$$

. . . . (q equations) . . . . . .

These equations contain all the unknown quantities (z) and are equal in number to q, the number of unknown quantities.

Hence, if the form of the function F were known, the solution of these equations would give the most probable values of z_1, z_2, etc.

The above equations, being perfectly general, hold true for all cases, so they must hold true for any special case. The form of F determined for the special case must therefore be the form of this function for all cases.

Consider n direct observations of equal weight on one unknown quantity z_1, the results of the measurements being M_1, M_2, ... M_n, and the residuals being denoted by v_1, v_2, ... v_n. The most probable value of z_1 is given by

$$z_1 = M_1 - v_1 = M_2 - v_2 = \cdots M_n - v_n.$$

Differentiating with respect to z_1,

$$1 = -\frac{\partial v_1}{\partial z_1} = -\frac{\partial v_2}{\partial z_1} = \cdots = -\frac{\partial v_n}{\partial z_1}. \qquad (a)$$

Substituting these values in Equa. [147], we obtain

$$F(v_1) + F(v_2) + \cdots + F(v_n) = 0. \qquad (b)$$

But in this special case (Art. 195),

$$v_1 + v_2 + \cdots + v_n = 0. \qquad (c)$$

Hence, if both Equas. (b) and (c) are true, F must signify multiplication by a constant; that is,

$$F(v) = cv. \qquad [148]$$

Substituting in Equas. [146] and [145],

$$\frac{\partial f(v)}{\partial z} = f(v) \cdot cv \frac{\partial v}{\partial z},$$

and

$$\frac{1}{f(v)} \cdot \frac{\partial f(v)}{\partial z} = cv \frac{\partial v}{\partial z}.$$

Integrating both members,

$$\log f(v) = \tfrac{1}{2} cv^2 + c'.$$

Therefore

$$f(v) = e^{(\frac{1}{2} cv^2 + c')}$$
$$= ke^{\frac{1}{2} cv^2}.$$

Substituting this in the equation of the curve of error ($y=f(v)$), we have

$$y = ke^{\frac{1}{2}cv^2}.$$

In reality y decreases as v increases; the exponent of e is therefore negative, and, since the constants may be combined, we have

$$y = ke^{-h^2v^2}, \qquad [149]$$

in which h^2 and k are constants depending upon the character of the observations. This equation expresses the law in accordance with which the residuals must be distributed in order to give a maximum value of P. If we replace v by x, the equation also shows the law governing the distribution of the actual errors.

It is important to note that the law governing the distribution of accidental errors holds true *in the long run;* in order to have a close agreement of the theory with the results actually observed it is essential that the number of observations should be very large. With a limited number of observations we should expect that the residuals would follow the law only approximately.

198. Computation of Most Probable Value.

From Equa. [143] we have seen that

$$P = f(v_1) \times f(v_2) \; . \; . \; . \times f(v_n)\,(dv)^n = \text{a maximum.} \qquad [150]$$

Applying Equa. [149], this becomes

$$P = k^n e^{-h^2(v_1^2 + v_2^2 + \; \cdot \cdot \; v_n^2)}(dv)^n = \text{a maximum.} \qquad [151]$$

It is evident that P is a maximum when

$$v_1^2 + v_2^2 + \cdot \cdot \cdot v_n^2 = \text{a minimum,} \qquad [152]$$

that is, when the sum of the *squares* of the residuals has its *least* value.

Equas. [147] express the conditions necessary to make P a maximum or to make the sum of the squares of the residuals a minimum. Since the function F means multiplication by a constant, Equas. [147] become

$$v_1 \frac{\partial v_1}{\partial z_1} + v_2 \frac{\partial v_2}{\partial z_1} + \qquad v_n \frac{\partial v_n}{\partial z_1} = 0.$$

$$v_1 \frac{\partial v_1}{\partial z_2} + v_2 \frac{\partial v_2}{\partial z_2} + \qquad v_n \frac{\partial v_n}{\partial z_2} = 0. \qquad [153]$$

$$v_1 \frac{\partial v_1}{\partial z_q} + v_2 \frac{\partial v_2}{\partial z_q} + \qquad \cdot\ v_n \frac{\partial v_n}{\partial z_q} = 0.$$

These equations are equal in number to the number, q, of unknown quantities, and their simultaneous solution gives the most probable values of the unknown quantities. They are usually called *Normal Equations*.

199. Weighted Observations.

If the observations are of different weights, each observation equation should be used (Art. 196) the number of times denoted by its weight. Hence, in forming the normal equations we should multiply each observation equation by the coefficient of the unknown *and by the weight of the equation*. The normal equations in this case are as follows:

$$\begin{aligned} &p_1 v_1 \frac{\partial v_1}{\partial z_1} + p_2 v_2 \frac{\partial v_2}{\partial z_1} + \\ &p_1 v_1 \frac{\partial v_1}{\partial z_2} + p_2 v_2 \frac{\partial v_2}{\partial z_2} + \qquad = 0. \\ &\qquad\qquad\qquad\qquad\qquad\quad = 0. \end{aligned} \qquad [154]$$

This same result will be obtained if we first multiply each observation equation by the square root of its weight. This shows that multiplying a set of equations by the square roots of their weights reduces them all to observations of weight unity (equal weights).

200. Relation between h and p.

If the n observations have weights p_1, p_2, . . . , and the constant h is h_1, h_2, . . . for these observations, then

$$\begin{aligned} P &= k_1 e^{-h_1^2 v_1^2} \cdot k_2 e^{-h_2^2 v_2^2} \ldots \\ &= k_1 k_2 \ldots k_n e^{-(h_1^2 v_1^2 + h_2^2 v_2^2 \ldots)} (dv)^n, \end{aligned} \qquad [155]$$

and $\qquad h_1^2 v_1^2 + h_2^2 v_2^2 + \cdots$ is to be a minimum. $\qquad [156]$

The conditions for this minimum are

$$h_1{}^2 v_1 \frac{\partial v_1}{\partial z_1} + h_2{}^2 v_2 \frac{\partial v_2}{\partial z_2} + \cdot \cdot$$

$$h_1{}^2 v_1 \frac{\partial v_1}{\partial z_1} + h_2{}^2 v_2 \frac{\partial v_2}{\partial z_2} + \cdot \cdot \quad = 0. \qquad [157]$$

Equas. [154] and [157] express the same conditions.

Hence $\qquad p_1 : p_2 : \cdot \cdot \cdot = h_1{}^2 : h_2{}^2 : \cdot \cdot \cdot ,$ [158]

showing that the weight of an observation varies as the square of the constant h for the observation. Consequently the more accurate the observation the greater the value of h.

Example. As an illustration of the manner of applying these equations to the computation of the most probable values of the unknowns, suppose that at a triangulation station O (Fig. 151), the angles have been measured as shown.

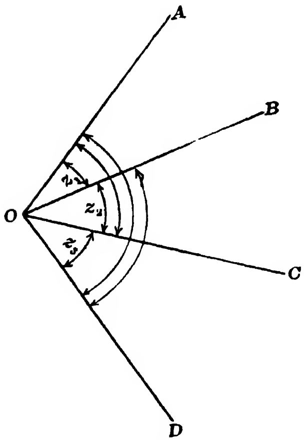


FIG. 151.

Denoting the most probable values of these angles by z_1, z_2, and z_3, the measurements are given by the following equations:

$$\begin{aligned} z_1 &= 31^\circ\ 10'\ 17''.0, \\ z_2 &= 40\ \ 50\ \ 10\ .0, \\ z_3 &= 42\ \ 10\ \ 19\ \ 7, \\ z_1 + z_2 &= 72\ \ 00\ \ 26\ .0, \\ z_1 + z_2 + z_3 &= 114\ \ 10\ \ 46\ .0, \\ z_2 + z_3 &= 83\ \ 00\ \ 30\ .2. \end{aligned}$$

Denoting by v_1, v_2, etc., the residuals of the different measurements, these may be written

$$\begin{aligned} z_1 - {} & 31^\circ\ 10'\ 17''.0 = v_1, \\ z_2 - {} & 40\ \ 50\ \ 10\ .0 = v_2, \\ z_3 - {} & 42\ \ 10\ \ 19\ .7 = v_3, \\ z_1 + z_2 - {} & 72\ \ 00\ \ 26\ .0 = v_4, \\ z_1 + z_2 + z_3 - {} & 114\ \ 10\ \ 46\ .0 = v_5, \\ z_2 + z_3 - {} & 83\ \ 00\ \ 30\ .2 = v_6, \end{aligned}$$

which are called *observation equations.*

If we apply equations (153), differentiating each v with respect to the three unknown quantities in succession and adding, we obtain the *normal equations*, —

$$\begin{aligned} 3\,z_1 + 2\,z_2 + \ \ z_3 - 217^\circ\ 21'\ 29''.0 &= 0, \\ 2\,z_1 + 4\,z_2 + 2\,z_3 - 310\ \ 01\ \ 52\ .2 &= 0, \\ z_1 + 2\,z_2 + 3\,z_3 - 239\ \ 21\ \ 35\ .9 &= 0. \end{aligned}$$

Solving these simultaneously, we obtain

$$z_1 = 31^\circ\ 10'\ 16''\ 45,$$
$$z_2 = 40\ \ 50\ \ 09\ .875,$$
$$z_3 = 42\ \ 10\ \ 19\ .90.$$

These are the most probable values of the angles.

201. Formation of the Normal Equations.

It should be observed that since the observation equations are linear in this case, the differential coefficients are equal to the numerical coefficients. Hence, to form the normal equations we may proceed as follows: *For each unknown, form a normal equation by multiplying each observation equation by the numerical coefficient of the unknown in that equation, adding these results and placing the sum equal to zero.* This rule is simply a statement in words of what is expressed in Formula [153] as applied to linear equations. If the observations are of different weights, the only change in the above rule is that each observation equation is multiplied by its weight as well as by the coefficient of the unknown.

In regard to the observation equations it should be understood that they are not like ordinary equations. They are often written, however, with zero in place of the v in the right-hand member. Observation equations cannot be multiplied by any number or combined with each other (except when forming normal equations); for if this is done, the *weight* of the observation is thereby changed.

202. Solution by Means of Corrections.

If the independent terms* in the observation equations are large, it will often save labor in the calculations if we place the unknown quantity Z_1 equal to an approximate value M_1 plus a correction z_1, $Z_2 = M_2 + z_2$, etc. Substituting these values in the original observation equations, we obtain a new set of equations in terms of the corrections and in which the independent terms will be small. By forming normal equations and solving

* The independent term in any equation is that term which does not contain any of the unknowns.

as before, we find the most probable values of the corrections. Adding these corrections to the approximate values, we find the most probable values of the unknown quantities themselves.

Example. In the example just solved, suppose we assume for the approximate values the results of the direct measurements, and let z_1, z_2, etc., represent the most probable corrections. Then the observation equations become

$$\begin{aligned} z_1 &= 0, \\ z_2 &= 0, \\ z_3 &= 0, \\ z_1 + z_2 + 1''.0 &= 0, \\ z_1 + z_2 + z_3 + 0\ .7 &= 0, \\ z_2 + z_3 - 0\ .5 &= 0. \end{aligned}$$

These equations may also be written with v_1, v_2, etc., instead of zeros in the right-hand member.

Forming the normal equations as before, we have

$$\begin{aligned} 3\,z_1 + 2\,z_2 + z_3 + 1''.7 &= 0, \\ 2\,z_1 + 4\,z_2 + 2\,z_3 + 1\ .2 &= 0, \\ z_1 + 2\,z_2 + 3\,z_3 + 0\ .2 &= 0. \end{aligned}$$

The solution of these equations gives

$$\begin{aligned} z_1 &= -0''.55, \\ z_2 &= -0\ \ 125, \\ z_3 &= +0\ .20, \end{aligned}$$

which, added to the values observed directly, give the same results as before.

203. Conditioned Observations.

If the quantities sought are not independent of each other, but are subject to certain conditions, the solution must be modified accordingly. Each observation gives rise to an observation equation, and each condition may be expressed by a condition equation. The solution may be effected by eliminating, between the two sets of equations, as many unknowns as there are equations of condition. From the remaining equations we may form the normal equations and solve for the most probable values of the unknowns. Substituting these values back in the original condition equations, we obtain the remaining unknowns.

Example. The three angles of a triangle are $A = 61°\ 07'\ 52''.00$, $B = 76°\ 50'\ 54''.00$, and $C = 42°\ 01'\ 12''.15$. The spherical excess is $02''.11$. The weights assigned to the measured angles are 3, 2, and 2, respectively. These angles are subject to the fixed relation $A + B + C = 180°\ 00'\ 02''\ 11$

Letting v_1, v_2, v_3 be the most probable corrections to the observed values, the observation equations are

$$\begin{aligned} v_1 &= v_1, & \text{wt. } 3 \\ v_2 &= v_2, & \text{" } 2 \\ v_3 &= v_3, & \text{" } 2 \end{aligned}$$

and the condition equation is

$$v_1 + v_2 + v_3 - 3''.96 = 0. \qquad (d)$$

Eliminating v_3, there remain

$$\begin{aligned} v_1 &= v_1, & \text{wt. } 3 \\ v_2 &= v_2, & \text{" } 2 \\ v_3 &= -v_1 - v_2 + 3''.96 & \text{" } 2 \end{aligned}$$

Forming the normal equations and solving,

$$\begin{aligned} v_1 &= +0''.99, \\ v_2 &= +1\ .485. \end{aligned}$$

Substituting these values in equation (*d*),

$$v_3 = +1''.485.$$

These corrections, added to the measured angles, give the adjusted angles, as follows:

$$\begin{aligned} A &= 61^\circ\ 07'\ 52''.99, \\ B &= 76\ \ \ 50\ \ 55\ \ .48, \\ C &= 42\ \ \ 01\ \ 13\ \ .64. \end{aligned}$$

Notice that the discrepancy is distributed inversely as the weights. This will always be the case when each unknown is directly observed, and there is but one equation of condition; that is, the correction to the first is

$$\frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{2} + \frac{1}{2}} \times +3''.96 = +0''.99,$$

and the correction to the second is

$$\frac{\frac{1}{2}}{\frac{1}{3} + \frac{1}{2} + \frac{1}{2}} \times +3''.96 = +1''.485.$$

The correction to the third is the same as the correction to the second.

204. Adjustment of Triangulation.

The adjustment of the angles of a triangulation net naturally divides itself into two parts: (1) the adjustment for the discrepancies arising at each station, and (2) the adjustment of the figure as a whole. According to theory these should all be adjusted simultaneously in order to obtain the most probable values of the angles. The usual practice, however, is to deal

with the two separately. The local, or station, adjustment is made first if the method of observing is such that a local adjustment is required. If the observations are made in accordance with the program given in Art. 45 (p. 88), no station adjustment is necessary. If the angles are measured by the repetition method and the horizon is closed, the error is distributed in inverse proportion to the weights (see Art. 203). If there are conditions existing among the angles, due to measuring sums of the different single angles, the adjustment may be effected by expressing these as condition equations and then forming normal equations and solving, as in the example, p. 366.

This method of making the local adjustment first is justified, not only on the ground of saving labor, but also because of the well-known fact that the most serious errors are those due to eccentricity of signal and instrument, phase of signal, refraction, etc., which do not appear to any large extent in the local adjustment but which do appear in the figure adjustment. If we compute the precision of angles from the discrepancies noted at each station, and then estimate from these values the error of closure to be expected in the triangle, we find that these are smaller than the errors of closure actually occurring, showing the presence of constant errors, which do not appear in the local adjustment.

205. Conditions in a Triangulation.

The geometric conditions connecting the angles in a net are of two classes: (1) those which express the relation among the angles of a triangle or other figure, and (2) those which express the relation existing among the sides of the figure. If we plot, for example, a quadrilateral figure, starting from one side as fixed, we shall find that if the sum of the angles in three of the triangles equals their theoretical sums, all sums in the other triangles will also (necessarily) equal their theoretical amounts, namely, $180° + \epsilon''$. This shows that of all the possible angle equations which might be written for this figure only three are really independent.

In order to determine the number of angle equations in any net, let s be the total number of stations, s_u the number of stations not occupied, l the total number of lines in the figure, and l_1 the number of lines sighted over in one direction only; then the number of angle equations in the figure is

$$l - l_1 - s + s_u + 1. \qquad [159]$$

In a triangle it is necessary that all stations should be occupied and that all lines should be sighted over in both directions, in order to have one angle equation, that is,

$$l - s + 1 = 3 - 3 + 1 = 1.$$

If a new station is added, it must be occupied and the two lines sighted over in both directions, in order to yield a new angle equation. If this is done, the quantity $l - s$ is increased by $2 - 1 = 1$. If a line is drawn between two stations already located, l is increased by 1 and there is a new angle equation corresponding. For each new line sighted in one direction only, l is increased by 1 and l_1 is increased by 1, so that the total is unchanged.

The number of side equations in a net may be estimated as follows: Starting with one line as fixed, it is evidently necessary to have two more sides in order to fix a third point. Hence, in order to plot a figure, we must have at least $2(s - 2)$ lines in addition to the base, that is, $2s - 3$ lines in all. Any additional lines used must conform to those already used, in order to give a perfect figure; hence the number of conditions giving rise to side equations will equal the number of superfluous lines, that is, $l - 2s + 3$, where l is the total number of lines and s is the number of stations. It should be observed that while the side equation is primarily a relation among the sides, it is also a relation among the sines of the angles, and this fact enables us to adjust the figure by altering the angles.

A check on the total number of conditions is obtained as follows: If n is the total number of lines, n' the number of lines observed in both directions, s the total number of stations, and s'

the number of occupied stations, then the number of conditions is

$$(n' - s' + 1) + (n - 2s + 3)$$

206. Adjustment of a Quadrilateral.

For any quadrilateral figure in which all of the (eight) angles have been measured there may be found three equations which express the condition that the triangles must all "close." There are more than three equations which may be formed; but if any three of these equations are satisfied, the others necessarily follow and hence are not independent. There will also be one side equation expressing the condition that the length of a side (AB), when computed from the opposite side (CD), is exactly the same, no matter which pair of triangles is employed in the computation.

In selecting the three angle equations we may take any three triangles and write an equation for each expressing the condition that the sum of the three angles equals $180° + \epsilon''$. It is advantageous in this case to avoid triangles having small angles. In selecting the side equation it is well, however, to select one involving small angles, so as to give large coefficients of the corrections. If the angle equations were also chosen so as to involve the small angles, the solution would be likely to prove unstable, on account of the equality of some of the coefficients.

A convenient method of writing a side equation is to select some point, called the *pole*, and write the three directions from it to the other stations in the order of azimuths. For example, taking the pole at A, Fig. 152, write first

$$AB \cdot AD \cdot AC.$$

Then from this write the ratios

$$\frac{AB}{AD} \cdot \frac{AD}{AC} \cdot \frac{AC}{AB},$$

the method of forming which is evident. If we now replace each line by the sine of the angle opposite to it in the triangle

which is indicated by the fraction, and place the whole equal to unity, we have

$$\frac{\sin ADB}{\sin ABD} \times \frac{\sin ACD}{\sin ADC} \times \frac{\sin ABC}{\sin ACB} \qquad [160]$$

It may be shown, by solving the different triangles and eliminating the sides, that this equation expresses the condition that

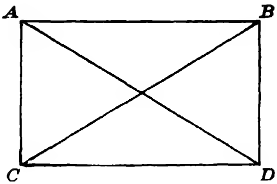


FIG. 152.

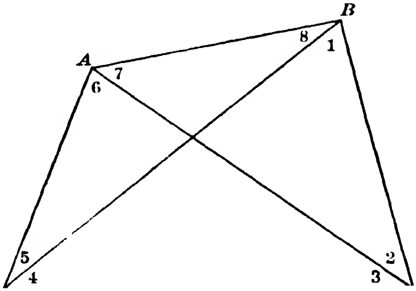


FIG. 153.

the length of AB as computed from CD is the same no matter which route is followed in the computation.

Problem. Prove by a direct solution of the triangles in Fig 152 that Equation [160] is true

Designating the angles by means of the numbers shown in Fig. 153, the equation becomes

$$\frac{\sin 2 \sin (4 + 5) \sin 8}{\sin (1 + 8) \sin 3 \sin 5} = 1. \qquad [161]$$

Before this equation can be used, however, it is practically necessary to reduce it to linear form, since an application of Equa. [153] to any but linear equations would be complicated.

Suppose our equation to be put in the general form

$$\frac{\sin (M_1 + v_1)}{\sin (M_2 + v_2)} \times \frac{\sin (M_3 + v_3)}{\sin (M_4 + v_4)} = 1, \qquad [162]$$

in which the angle is written as an approximate value M plus a small correction v. Taking logs of both members and then applying Taylor's theorem, we have, neglecting squares and higher powers,

$$\log \sin M_1 + \frac{\partial}{\partial M_1}(\log \sin M_1)\, v_1 +$$
$$\left(\log \sin M_2 + \frac{\partial}{\partial M_2}(\log \sin M_2)\, v_2 + \cdot \cdot \cdot\right) = 0. \quad [163]$$

The quantity $\frac{\partial}{\partial M_1}(\log \sin M_1)$ is the variation per 1″ in a table of log sines, the correction v being in seconds. Hence, placing $\delta_1 = \frac{\partial}{\partial M_1}(\log \sin M_1)$, etc., we have

$$\delta_1 v_1 - \delta_2 v_2 + \delta_3 v_3 - \delta_4 v_4 + \cdot \cdot \cdot$$
$$+ \log \sin M_1 - \log \sin M_2 + \cdot \cdot \cdot = 0. \quad [164]$$

The algebraic sum of the log sines represents the amount by which they fail to satisfy the condition equation. Placing this sum equal to l, the side equation given above becomes

$$\delta_2 v_2 + \delta_{4+5} v_{4+5} + \delta_8 v_8 - (\delta_{1+8} v_{1+8} + \delta_3 v_3 + \delta_5 v_5) + l = 0. \quad [165]$$

Example. Let us suppose that the measured angles are (Fig. 153),

1.	61°	07′	52″	.00
2.	38	28	34	.90
3.	38	22	19	.10
4.	42	01	12	.15
5.	29	14	32	.85
6.	70	21	59	.20
7.	49	26	21	.85
8.	30	57	07	.10

These angles are supposed to have been adjusted for local conditions.

To form the angle equations, take the triangles *ABD*, *ADC*, and *ABC* for which the values of the spherical excess are 1″.36, 1″.77 and 1″.02, respectively. The computation is shown in tabular form as follows:

1 + 8	92° 04′ 59″.10
2	38 28 34 00
7	49 26 21 .85
	179 59 55 85
	180 00 01 .36
	+5″ 51
3	38° 22′ 10″.10
4 + 5	71 15 45 .00
6	70 21 59 20
	180 00 03 .30
	180 00 01 77
	−1″ 53
5	29° 14′ 32″.85
6	70 21 59 .20
7	49 26 21 85
8	30 57 07 10
	180 00 01 00
	180 00 01 02
	+0″ 02

This gives for the three angle equations

$$(1 + 8) + 2 + 7 = 180° 00' 01''.36,$$
$$3 + (4 + 5) + 6 = 180\ 00\ 01\ .77,$$
$$5 + 6 + 7 + 8 = 180\ 00\ 01\ .02,$$

or, written as corrections,

$$v_{1+8} + v_2 + v_7 - 5.51 = 0,$$
$$v_3 + v_{4+5} + v_6 + 1.53 = 0,$$
$$v_5 + v_6 + v_7 + v_8 - 0.02 = 0,$$

the corrections being the same as the residuals in this case.

To form the side equation, take the pole at A. Then we have

$$\frac{AB}{AD}\cdot\frac{AD}{AC}\cdot\frac{AC}{AB},$$

giving

$$\frac{\cdot}{\sin(1+8)}\cdot\frac{\sin(4+5)}{\sin 3}\cdot\frac{\sin 8}{\sin 5} = 1,$$

$$\log\sin 2 + \log\sin(4+5) + \log\sin 8 - \log\sin(1+8) - \log\sin 3 - \log\sin 5 = 0.$$

The computation of the constant term of this equation is given in the following table The log sines of those angles appearing in the numerator, together with their diff. for 1″ (in units of the 6th place of decimals) are placed in the left-hand column, and those in the denominator are placed in the right-hand column. The constant l is the difference in the sums of the log sines.

Angle	log sine (+).	Diff. 1″	Angle.	log sine (−).	Diff. 1″.
2	9.7939242	+2.65	1+8	9.9997129	−0.08
4+5	9.9763501	+0.72	3	9.7929268	+2.66
8	9.7112329	+3.51	5	9.6888702	+3.76
	9.4815072			9.4815099	
				72	
				−27	

Therefore $\qquad l = -2.7.$

The side equation becomes

$$2.65\, v_2 + 0.72\, v_{4+5} + 3.51\, v_8 + 0.08\, v_{1+8} - 2.66\, v_3 - 3.76\, v_5 - 2.7 = 0,$$

or, combining the v_5 and the v_8 terms,

$$+.08\, v_1 + 2.65\, v_2 - 2.66\, v_3 + .72\, v_4 - 3.04\, v_5 + 3.59\, v_8 - 2.70 = 0.$$

207. Solution by Direct Elimination.

The observations are all direct and equal in number to the number of unknown angles, eight. The four condition equations have just been stated. We might, therefore, proceed as in Art. 203, that is, eliminate four of the unknowns between these two sets of equations and then form normal equations and solve for the remaining four unknowns. Substitution back in the condition equations will enable us to find the four unknowns that were first eliminated.

The foregoing process is not the shortest one available, nor is it the best from the standpoint of accuracy. The labor involved in eliminating the four unknowns, that is, expressing all of the residuals in terms of four selected unknowns, is often considerable. Furthermore the coefficients are often such that an accurate determination of the unknowns is difficult. The method of correlatives (Art. 210, p. 377) is generally more suitable for such adjustments.

208. Gauss's Method of Substitution.

In solving a large number of equations simultaneously it is convenient to use some definite system of eliminating the unknowns, in order to avoid labor and the danger of mistakes.

Let us suppose that the observation equations are of the form

$$a_1x + b_1y + c_1z + l_1 = v_1,$$
$$a_2x + b_2y + c_2z + l_2 = v_2,$$

and that the normal equations are represented by

$$\left.\begin{aligned} [aa]x + [ab]y + [ac]z + [al] &= 0, \\ [ab]x + [bb]y + [bc]z + [bl] &= 0, \\ [ac]x + [bc]y + [cc]z + [cl] &= 0, \end{aligned}\right\} \qquad [166]$$

in which the brackets indicate the sum of all the terms found by multiplying the numerical coefficients according to the rule on p. 365.

If the first normal equation be divided by $[aa]$ and solved for x, the result is

$$x = -\frac{[ab]}{[aa]}y - \frac{[ac]}{[aa]}z - \frac{[al]}{[aa]}.$$

Substituting this in the second equation, we have

$$\left([bb] - \frac{[ab]}{[aa]}[ab]\right)y + \left([bc] - \frac{[ac]}{[aa]}[ab]\right)z + \left([bl] - \frac{[al]}{[aa]}[ab]\right) = 0 \quad [167]$$

This is usually abbreviated

$$[bb \cdot 1]\,y + [bc \cdot 1]\,z + [bl \cdot 1] = 0. \qquad [168]$$

Substituting this in the third equation, we have

$$[bc \cdot 1]\,y + [cc \cdot 1]\,z + [cl \cdot 1] = 0. \qquad [169]$$

These two equations, [168] and [169], are called the "first reduced normal equations."

Solving [168] for y,

$$y = -\frac{[bc \cdot 1]}{[bb \cdot 1]}z - \frac{[bl \cdot 1]}{[bb \cdot 1]};$$

whence $$[cc \cdot 2]\,z + [cl \cdot 2] = 0, \qquad [170]$$

in which
$$[cc \cdot 2] = [cc \cdot 1] - \frac{[bc \cdot 1]}{[bb \cdot 1]}[bc \cdot 1]$$

and
$$[cl \cdot 2] = [cl \cdot 1] - \frac{[bc \cdot 1]}{[bb \cdot 1]}[bl \cdot 1].$$

The solution of [170] gives the value of z. By substituting this in [168] and [169] the value of y may be found. Finally, from [166] the value of x may be found.

An inspection of [166] will show that all coefficients below and to the left of a diagonal drawn from the x term of the first equation to the z term of the third equation are duplicates of the others. These may be omitted in writing the equations.

Doolittle's Abridged Method.

In carrying out a solution by the Gauss method it will be observed that certain steps are taken which are not essential to the final result. In eliminating x from the equations the coefficients of x are purposely made to add up to zero, so it is unnecessary to write these coefficients and they may be omitted in the solution. Similarly, in eliminating y from the first reduced normal equations the y coefficients need not be entered. Furthermore the Gauss method calls for the addition of two equations (in the second elimination) to form a third, and then this third is later combined with a fourth, but is not used again. In the short method the first, second and fourth equations are combined directly and the third is not written at all. In each successive elimination the number of equations that are added at one step is increased. This method will be illustrated in detail in the examples in Art. 210. An explanation of the method will be found in Appendix 8 of the Report of the Superintendent of the U. S. Coast and Geodetic Survey for 1878.

209. Checks on the Solution.

In practice it would not be advisable to proceed in the solution of a large number of equations without some safeguard against mistakes of computation. A valuable check consists in adding to the normal equations an extra term which is merely the sum

of all the coefficients of v_1, v_2, etc., and treating this term like any other term of the equation. This is illustrated later in the example on pp. 381 and 388.

210. Method of Correlatives.

When there are many condition equations, the method of substitution is likely to prove laborious. If, as is usually the case in triangulation, the observations are direct and equal in number to the number of unknowns, the "Method of Correlatives" will be found preferable. By this method we eliminate one unknown for each condition equation, employing for this purpose the method of undetermined multipliers.

Suppose that we have made m direct observations, M_1, $M_2, \cdot \cdot \cdot, M_m$, of m different quantities, of which the most probable values are

$$z_1 = M_1 + v_1, \quad z_2 = M_2 + v_2, \ldots, \quad z_m = M_m + v_m.$$

Let these m unknowns be connected by the following n condition equations:

$$\begin{aligned} a_1v_1 + a_2v_2 \ldots a_mv_m + l_1 &= 0, \\ b_1v_1 + b_2v_2 \ldots b_mv_m + l_2 &= 0, \end{aligned} \qquad [171]$$

the a's being the coefficients in the first equation, the b's those of the second, etc. The quantities l_1, l_2, etc., represent the amounts by which the observations fail to satisfy the condition equations. If the original condition equations are not linear in form, they must be put in linear form by a method similar to that given on pp. 371–72.

Since the most probable values of the v's are to be found, we must have

$$v_1^2 + v_2^2 + \cdot \cdot \cdot = \text{a minimum}, \qquad [172]$$

or

$$v_1\,dv_1 + v_2\,dv_2 + \cdot \cdot \cdot = 0 \qquad [172a]$$

for all possible simultaneous values of dv_1, dv_2, etc.

Hence it must hold true for the equations

$$\begin{aligned} a_1\,dv_1 + a_2\,dv_2 + \cdot \cdot \cdot &= 0, \\ b_1\,dv_1 + b_2\,dv_2 + \cdot \cdot \cdot &= 0, \end{aligned} \qquad [173]$$

obtained by differentiating [171] because the values of v must satisfy the condition equations [171] as well as [172] or [172*a*]. The number of these equations is n. The number of terms in [172] is m, m being greater than n. Let the first equation in [173] be multiplied by k_1, the second by k_2, etc., and Equa. [172] by -1. The products are then added, giving

$$\begin{aligned} &(a_1k_1 + b_1k_2 + \cdot \cdot \cdot - v_1)\,dv_1 \\ &\qquad + (a_2k_1 + b_2k_2 + \cdot \cdot \cdot - v_2)\,dv_2 + \cdot \cdot \cdot = 0. \end{aligned} \qquad [174]$$

The k's are to be so determined that this equation will hold true. This equation will be satisfied if the coefficient of each differential in it is placed equal to zero, that is, if

$$\left.\begin{aligned} k_1a_1 + k_2b_1 + \cdot \cdot \cdot k_nl_1 &= v_1, \\ k_1a_2 + k_2b_2 + \cdot \cdot \cdot k_nl_2 &= v_2, \end{aligned}\right\} \qquad [175]$$

Substituting these values of v_1, v_2, etc., from Equa. [175] in Equa. [171], we obtain

$$\begin{aligned} k_1\,[aa] + k_2\,[ab] + \cdot \cdot \cdot k_n\,[al] + l_1 &= 0, \\ k_1\,[ab] + k_2\,[bb] + \cdot \cdot \cdot k_n\,[bl] + l_2 &= 0, \end{aligned} \qquad [176]$$

The solution of these equations gives the values of k_1, k_2, k_3, etc., which are the correlatives of the condition equations. By substituting these values in Equa. [175] the v's are found. Since the form of Equa. [176] is the same as that of normal equations, it is evident that they may be solved by the method of substitution.

In case the observations are of different weight, the minimum equation would be

$$p_1v_1^2 + p_2v_2^2 + \cdot \cdot \cdot p_mv_m^2 = \text{a minimum}, \qquad [177]$$

and the other equations would be modified accordingly.

Equations [175] and [176] may also be derived by the following method. If we multiply the condition equations [171] in succession by $-2\,k_1$, $-2\,k_2$, etc., we have

$$-2\,k_1a_1v_1 - 2\,k_1a_2v_2 \;.\;.\;. = 0$$
$$-2\,k_2b_1v_1 - 2\,k_2b_2v_2 \;.\;.\;. = 0.$$

Adding these to [172] and grouping the coefficients of the different v's, we have

$$\begin{aligned} &v_1^2 - 2\,v_1\,(a_1k_1 + b_1k_2 + \cdots) \\ &+v_2^2 - 2\,v_2\,(a_2k_1 + b_2k_2 + \cdots) \\ &+ \cdots \qquad = \text{a minimum.} \end{aligned}$$

For a minimum the derivative of this expression with respect to each separate v must be zero. Therefore,

$$2\,v_1 - 2\,(a_1k_1 + b_1k_2 + \cdots) = 0$$
$$2\,v_2 - 2\,(a_2k_1 + b_2k_2 + \cdots) = 0$$

Solving for the v's,

$$\left.\begin{aligned} v_1 &= a_1k_1 + b_1k_2 + \cdots \\ v_2 &= a_2k_1 + b_2k_2 + \cdots \end{aligned}\right\} \qquad [175]$$

which express the residuals in terms of the correlatives.

Substituting these values of v in the condition equation [171] we obtain the final equations [176].

Example. As an illustration of the method of correlatives and also the method of (abridged) substitution we will adjust the quadrilateral given on page 372. The four condition equations are,

$$\text{Angle equations}\quad \begin{aligned} v_1 + v_8 + v_2 + v_7 - 5.51 &= 0, \\ v_3 + v_4 + v_5 + v_6 + 1.53 &\cdot 0. \\ v_5 + v_6 + v_7 + v_8 - 0.02 &= 0. \end{aligned}$$

Side equation,

$$+0.08\,v_1 + 2.65\,v_2 - 2.66\,v_3 + 0.72\,v_4 - 3.04\,v_5 + 3.59\,v_8 - 2.70 = 0.$$

In order to facilitate the formation of the coefficients in the so-called "correlate" equations, $[aa]$, $[ab]$, etc., the coefficients of the four conditions will first be tabulated.

CONDITION EQUATIONS

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	Const.
a	+1	+1					+1	+1	−5.51
b			+1	+1	+1	+1			+1.53
c					+1	+1	+1	+1	−0.02
d	+0.08	+2.65	−2.66	+0.72	−3.04			+3.59	−2.70

Forming the coefficients and substituting in the correlate equations [176] we have the following equations for determining the k's, which are often called Normal Equations.

$$\begin{aligned}
4\,k_1 + 0\,k_2 + 2\,k_3 + 6.32\,k_4 - 5.51 &= 0,\\
0\,k_1 + 4\,k_2 + 2\,k_3 - 4.98\,k_4 + 1.53 &= 0,\\
2\,k_1 + 2\,k_2 + 4\,k_3 + 0.55\,k_4 - 0.02 &= 0,\\
+6.32\,k_1 - 4.98\,k_2 + 0.55\,k_3 + 36.75\,k_4 - 2.70 &= 0.
\end{aligned}$$

All those terms below and to the left of the zig-zag line are seen to be duplicates of others above and to the right; that is, the coefficients in the first column are the same as those in the first row; those in the second column as those of the second row, and so on. The coefficients of these equations are next tabulated in the form used for the solution. The terms to the left of the diagonal are omitted. For reasons which will be seen later on the first equation (p. 381) is assigned the Roman numeral I and the next three are assigned the Arabic numerals, 2, 3, 4.

The first step in the solution is to eliminate k_1 between Equas. I and 2. This is done by transferring Equa. 2 below and writing beneath it Equa. I multiplied by the coefficient of k_2 and divided by the coefficient of k_1, and with the sign reversed; the operation is indicated in the column at the right. This equation (not numbered) is then added to Equa. 2. In this case the equation added happens to have only zeros for coefficients. The result is Equa. II, the first reduced normal equation. In a similar manner we add to Equa. 3 the result obtained by multiplying I by the coefficient of k_3, dividing by the coefficient of k_1 with sign

changed. We also add to 3 the result obtained by multiplying II by the coefficient of k_3, dividing by the coefficient of k_2, and changing the sign. The sum of these three gives III, which contains only k_3 and k_4 and which is the second reduced normal equation The last step is similar, giving Equa. IV, which contains only k_4 and may be solved for this correlative.

	k_1	k_2	k_3	k_4	Const.	Check	
1	4	0	2	+ 6 32	− 5 51	+ 6 81	
2		4	2	− 4 98	+ 1 53	+ 2 55	
3			4	+ 0 55	− 0 02	+ 8 53	
4				+36 75	− 2 70	+35 94	
2		4	2	− 4 98	+ 1 53	+ 2 55	
		0	0	0	0	0	I × $-\frac{0}{4}$
II		4	2	− 4 98	+ 1 53	+ 2 55 ✓	
3			4	+ 0 55	− 0 02	+ 8 53	
			−1	− 3 16	+ 2 755	− 3 405	I × $-\frac{2}{4}$
			−1	+ 2 49	− 0 765	− 1 275	II × $-\frac{2}{4}$
III			2	− 0 12	+ 1 97	+ 3 85 ✓	
4				+36 750	− 2.70	+35 94	
				− 9 986	+ 8 706	−10 76	I × $-\frac{6.32}{4}$
				− 6 200	+ 1 905	+ 3.175	II × $+\frac{4.98}{4}$
				− 0 007	+ 0 118	+ 0 231	III × $+\frac{.12}{2}$
IV				+20 557	+ 8 029	+28 586 ✓	

The general process to be followed in writing any set of these partial equations may be seen by examining one group, as for example, those to be added to 3 to obtain III. The first coefficient appearing in Equa. 3 is 4. Following this column to the top we find 2, which is the numerator of the multiplier of I. The denominator is the first coefficient in I, or 4. In forming the next equation follow up the same column until we reach Equa. II, the numerator of this multiplier being 2; the denominator is the first coefficient of Equa. II, which is 4. The same system will be seen to give the multipliers for the last group.

The check term in any equation is the sum of all of the other coefficients in that equation. Every operation performed on the coefficients of an equation is also performed on the check term. In all complete equations, therefore, like II, III, and IV, the check term should still equal the sum of the coefficients. In the other equations it should not, because some terms in these equations have been omitted in writing. In summing the coefficients in any equation, say Equa. 3, start from the top of the k_3 column and follow down to the line of Equa. 3, then to the right. The coefficients for this equation are 2, 2, 4, + 0.55, −0.02; the sum is +8.53.

From the four equations IV, III, II, and I we obtain the values of k_4, k_3, k_2 and k_1 as follows:

$$k_4 = \frac{-8.029}{20.557} = -0.3906$$

$$k_3 = +.12 \times -.3906 - 1.97 = -1.008+$$

$$k_2 = \frac{-2 \times -1.008 + 4.98 \times -.3906 - 1.53}{4} = -.365$$

$$k_1 = \frac{-2 \times -1.008 - 6.32 \times -.3906 + 5.51}{4} = +2.50.$$

In finding the values of v_1, v_2, etc., from the equations corresponding to those given in Equa. [175] it will be convenient to tabulate the correlatives as follows:

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
k_1	+2.50	+2 50					+2.50	+2 50
k_2			−.365	− 365	− .365	− 365		
k_3					−1 008	−1 008	−1.008	−1 008
k_4	−.031	−1 036	+1.040	−.281	+1 189			−1 404
	″	″	″	″	″	″	″	″
Corr.	+2 47	+1.46	+0.675	−0.65	−0.18	−1.37	+1.49	+0.09

In this particular example the coefficients of k_1, k_2, and k_3 are

all unity; in the last line each k_4 is multiplied by the corresponding coefficient in the side equation.

Applying these corrections to the original angles we obtain,

1.	61°	07′	52″.00 + 2″.47	=	54″.47	
	38	28	34 .90 + 1 .46	=	36 .36	
3.	38	22	19 .10 + 0 .675	=	19 .78	
4.	42	01	12 .15 − 0 .65	=	11 .50	
5.	29	14	32 .85 − 0 18	=	32 .66	
6.	70	21	59 .20 − 1 .37	=	57 .83	
7.	49	26	21 .85 + 1 .49	=	23 .34	
	30	57	07 .10 + 0 .09	=	07 .19	

To test these results the v's are substituted in the condition equations.

ANGLE EQUATIONS

(1)	+2″.47	
(2)	+1 .46	
(7)	+1 .49	
(8)	+0 .09	
sum =	+5 51	Check
(3)	+0″.675	
(4)	−0 .65	
(5)	−0 18	
(6)	−1 37	
sum =	−1 525	Check
(5)	−0″ 18	
(6)	−1 .37	
(7)	+1 .49	
(8)	+0 .09	
sum =	+0 03	Check

SIDE EQUATION

$$+0.08\,v_1 + 2.65\,v_2 - 2.66\,v_3 + 0.72\,v_4 - 3.04\,v_5 + 3.59\,v_8 - 2.70 = -0.027$$

The side equation may also be tested by taking out the log sines of the corrected angles.

Angle	log sin (+)	diff 1″	Angle	log sin (−)	diff. 1″
2	9.793 9281	+2 65	1+8	9.999 7127	−0 08
4+5	9.976 3495	+0.72	3	9 792 9286	+2.66
8	9.711 2332	+3 51	5	9 688 8695	+3.76
	9.481 5108			9.481 5108	

The test of the sums of the angles in the triangles is as follows:

ANGLE EQUATIONS

(1)	54″.47
(2)	36 .36
(7)	23 .34
(8)	07 .19
sum =	1″.36 = e
	Check

(3)	19″.78
(4)	11 .50
(5)	32 .66
(6)	57 .83
sum =	1″.77 = e
	Check

(5)	32″.66
(6)	57 .83
(7)	23 .34
(8)	07 .19
sum =	1″.02 = e
	Check

211. Method of Directions.

The method of correcting the directions instead of the angles is particularly applicable when the measurements have been taken by the method of directions, Art. 43. In the United States Coast Survey office it is the usual practice to employ this method of adjusting, whether the observations were made by the direction method or by the method of repetition.

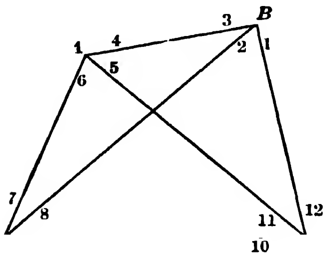


FIG. 154.

In the quadrilateral adjusted in Art. 210, let us denote the directions by the numbers 1 to 12 (Fig. 154) and the corrections to those directions by the same numbers, (1), (2), etc., enclosed in parentheses. Each angle is expressed as the difference of two directions; that is, the angle $-4 + 5$ means the angle between

the directions marked 4 and 5. The four condition equations are the same as before except as to the change in notation.

$$\text{Angle equations}\quad \begin{aligned} &-(4)+(5)-(1)+(3)-(11)+(12)-5.51 = 0. \\ &-(5)+(6)-(7)+(9)-(10)+(11)+1.53 = 0. \\ &-(2)+(3)-(7)+(8)-(4)+(6)-0.02 = 0. \end{aligned}$$

Side equation,

$$\begin{aligned} &-5.31\,(11) + 2.65\,(12) + 3.04\,(7) + 0.72\,(9) - 3.51\,(2) \\ &+3.59\,(3) - 0.08\,(1) + 2.66\,(10) - 3.76\,(8) - 2.7 = 0. \end{aligned}$$

If *CD* were a fixed line obtained by a previous adjustment, the corrections (9) and (10) would be omitted. The angle equations could be simplified in this case by selecting two equations which involve angles depending upon those two directions.

The first table for the coefficients of the corrections is given below.

Direction	*a*	*b*	*c*	*d*
1	−1			−0.08
2			−1	−3 51
3	+1		+1	+3.59
4	−1		−1	
5	+1	−1		
6		+1	+1	
7		−1	−1	+3.04
8			+1	−3 76
9		+1		+0 72
10		−1		+2 66
11	−1	+1		−5 31
12	+1			+2 65

The remainder of the work, that is, the calculation of coefficients $\sum aa$, $\sum ab$, etc., and the solution of the numerical equations, is carried out as in the preceding example (Art. 210). The solution of the normal equations gives the corrections to the directions. The correction to any angle is the difference of the corrections to the directions of its sides.

212. Adjusting New Triangulation to Points already Adjusted.

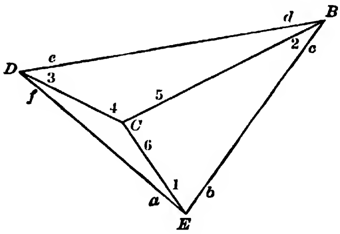


Fig. 155.

In the quadrilateral shown in Fig. 155 the triangle *BDE* is supposed to have been previously adjusted. Point *C* is determined by the directions 1, 2, and 3 in connection with the directions along the sides of the fixed triangle, and also by directions 4, 5, and 6. The directions to be found are 1, 2, 3, 4, 5, and 6. The directions as taken from the field-notes are as follows:

Point sighted.	Direction after local adjustment.	Corrected seconds.
	At *C*	
	° ′ ″	
D	0 00 00 00	
B	123 49 24 97	
E	207 52 33 50	
	At *D*	
	° ′ ″	″
A	0 00 00.00	00 67
C	296 57 55.83	
E	311 12 14.48	12 69
B	258 27 57.39	57 18
	At *E*	
	° ′ ″	″
D	0 00 00.00	01.32
C	13 38 27 54	
B	81 28 43 98	43.05
	At *B*	
	° ′ ″	″
F	0 00 00.00	01 06
E	122 32 11.29	12.56
C	150 38 41.62	
D	168 19 14.81	15 48

In taking directions from this table, the corrected seconds should be used whenever an adjustment has been made.

The number of angle equations in the figure is $l - s + 1$, or $6 - 4 + 1 = 3$. The number of side equations is $l - 2s + 3$, or $6 - 8 + 3 = 1$. Since, however, the exterior triangle is already adjusted, there will be but two angle equations needed in the adjustment. For these two angle equations let us take the triangles DCE and BEC;

then $\quad -(a) + (1) - (3) + (f) - (6) + (4) + 9''.56 = 0$

and $\quad -(1) + (b) - (5) + (6) - (c) + (2) - 6''.08 = 0.$

But since the exterior lines are not to be changed, (a), (f), (b), and (c) are all zero.

The absolute terms in the angle equations are found as follows:

$-(a) + (1)$	13° 38′ 26″.22
$-(3) + (f)$	14 14 16 86
$-(6) + (4)$	152 07 26 50
	180 00 09 .58
	180 00 00 02
	−09″.56
$-(1) + (b)$	67° 59′ 15″ 51
$-(5) + (6)$	84 03 08 53
$-(c) + (2)$	28 06 29 .06
	179 59 53 .10
	180 00 00 08
	+6″.08

For the side equation take the pole at C.

$$\frac{\sin(-(2) + (d))}{\sin(-(e) + (3))} \cdot \frac{\sin(-(1) + (b))}{\sin(-(c) + (2))} \cdot \frac{\sin(-(3) + (f))}{\sin(-(a) + (1))} = 1.$$

Tabulating the log sines,

		log sin (+)	diff 1″
$-(2) + (d)$	17° 40′ 33″.86	9.4823521	+66.1
$-(1) + (b)$	67 50 15 .51	9.9666666	+ 8.6
$-(3) + (f)$	14 14 16 .86	9.3908478	+83.0
		8.8398665	
$-(e) + (3)$	38° 29′ 58″.65	9.7941460	+26.5
$-(c) + (2)$	28 06 29 .06	9.6731464	+39.5
$-(a) + (1)$	13 38 26 .22	9.3726010	+86.7
		8.8398934	
		8665	
	constant =	−269	

The side equation is therefore

$$+6.61 \times -(2) + 0.86 \times -(1) + 8.30 \times -(3)$$
$$-2.65 \times (3) - 3.95 \times (2) - 8.67 \times (1) - 26.9 = 0.$$

Carrying out the same process as outlined in Art. 210, we have the following:

CONDITION EQUATIONS

	v_1	v_2	v_3	v_4	v_5	v_6	Const.
a	$+1$		-1	$+1$		-1	$+9.56$
b	-1	$+1$			-1	$+1$	-6.98
c	-9.53	$-10\ 56$	$-10\ 95$				$-26\ 9$

Forming the correlate equations and tabulating as in Art. 210 we find for the equations determining k_1, k_2, and k_3 the three numbered I, 2, and 3 in the following table.

	k_1	k_2	k_3	Const	Check	
I	$+4$	-2	$+1.42$	$+9.56$	$+12.98$	
2		$+4$	-1.03	-6.98	-6.01	
3			$+322\ 24$	-26.9	$+295.73$	
2		$+4$	-1.03	$-6\ 98$	-6.01	
		-1	$+0\ 71$	$+4\ 78$	$+6\ 49$	$\mathrm{I} \times -\frac{-2}{4}$
II		$+3$	$-0\ 32$	$-2\ 20$	$+0\ 48$	
3			$+322.24$	$-26\ 9$	$+295.73$	
			-50	-3.39	-4.61	$\mathrm{I} \times -\frac{1.42}{4}$
			-03	$-.24$	$+05$	$\mathrm{II} \times -\frac{-.32}{3}$
III			$+321.71$	-30.53	$+291.17$	

From I, II and III we find

$$k_3 \quad \frac{30.53}{321.70} = +.0949$$

$$k_2 \quad \frac{.32 \times .0949 + 2.20}{3} = +.7433$$

$$k_1 \quad \frac{2 \times .7433 - 1.42 \times .0949 - 9.56}{4} \qquad -2.052.$$

Tabulating the corrections, we have

	v_1	v_2	v_3	v_4	v_5	v_6
k_1	−2.052		+2.052	−2.052		+2.052
k_2	−0.743	+0.743			−0.743	−0.743
k_3	−0.904	−1.002	−1.039			
Corr.	−3.699	−0.262	+1.013	−2.052	−0.743	+2.795

Applying these corrections to the observed directions, we have the final adjusted values.

Direction No	Observed directions	Correction	Corrected seconds
	° ′ ″	″	″
4	0 00 00.00	−2.05	57.95
5	123 49 24.97	−0.74	24.23
6	207 52 33.50	+2.80	36.30
1	13 38 27.54	−3.70	23.84
2	150 38 41.62	−0.26	41.36
3	296 57 55.83	+1.01	56.84

212a. Adjusting Traverses to Triangulation.

Whenever a traverse starts from a fixed triangulation point and closes on another the angles of the traverse, or the sides, or both, must receive corrections so that the final results are in agreement with the fixed triangulation. The method of making such an adjustment as usually practiced is more or less arbitrary in character and depends upon the sort of errors of closure revealed by the particular traverse in question. There is no general rule which applies to all cases. The amount of error that is to be attributed to lengths and to angles differs according to circumstances. In Special Publication No. 137 of the U. S. Coast and Geodetic Survey will be found a discussion of the methods used for first-order traverse of that survey.

The traverse is ordinarily treated as a plane figure because the area covered is not great, except in the case of some first-

order traverses. When the assumption of a plane figure can be made the only effect of curvature that must be provided for is the convergence of the meridians. This may be avoided by referring all bearings or azimuths to the initial meridian (or any other selected meridian). With such azimuths the entire traverse may be balanced according to the usual methods of plane surveying so that it will have the required length and direction between its terminal points.

212b. Adjustment of Level Circuits.

Whenever lines of levels have been run in circuits the resulting differences in elevation between bench marks will require an adjustment in order to remove any inconsistencies in these differences. The adjustment is made by writing a condition equation for each circuit closure and solving by the method of correlatives. (Art. 210.) In making this adjustment, it is advisable to draw a sketch showing all the lines that have been run; from this sketch the condition equations may be formed. On each line of the sketch are written (1) the distance between bench marks, (2) the observed difference in elevation, with its algebraic sign, (3) an arrow showing the direction in which the difference in elevation applies, and (4) a reference number. Inside of each circuit is written the amount of the error of closure of that circuit and also a reference number or letter.

The errors of leveling are treated as accidental and the observed differences of elevation are therefore given weights which are inversely proportional to the lengths of the lines. This is done by dividing some arbitrarily chosen number by the different distances.

In writing the condition equations it should be noted that there are as many equations as there are superfluous lines, that is, for each line which runs to a bench already established there will be a condition. The number of conditions will equal the number of lines $-$ the number of bench marks $+1$. When forming these equations proceed around each circuit clockwise from start to finish. If any difference in elevation has an arrow

pointing in the opposite direction from this, the sign of the corresponding difference in elevation must be reversed.

When the condition equations have been stated then Equas. [176] for the correlatives may be formed and solved. Substituting the resulting values of the correlatives in Equas. [175] the corrections themselves are found. Adding these corrections to the observed differences in elevation the final adjusted differences in elevation are obtained. After the completion of the adjustment, the elevation of any bench mark as found by following any route should be the same as that found by following any other route.

It is possible to make this adjustment by assuming approximate elevations for the various bench marks and solving for corrections to these assumed elevations, instead of dealing with the differences in elevation for each line.

If the elevations of more than one bench mark are to be held fixed in the adjustment extra equations must be introduced for this purpose.

For examples of adjustment and details of the method see Leland's *Practical Least Squares*, p. 64, and U. S. Coast and Geodetic Survey Special Publ. No. 140, p. 53.

213. The Precision Measures.

Referring to the equation of the curve of error, Art. 197,

$$y = ke^{-h^2x^2}, \qquad [149]$$

we see that there are two constants to be determined for any particular set of observations. These two constants are not independent, however, as will be shown. The total area between the curve and the X axis was taken equal to unity; therefore

$$k\int_{-\infty}^{\infty} e^{-h^2x^2}\,dx = 1,$$

or

$$k\int_{0}^{\infty} e^{-h^2x^2}\,dx = \tfrac{1}{2},$$

from which

$$\int_{0}^{\infty} e^{-h^2x^2}\,h\,dx = \frac{h}{2\,k}.$$

In order to integrate this expression let $t = hx$ and $dt = h\,dx$.

Then
$$\int_0^\infty e^{-t^2}\,dt = \int_0^\infty e^{-h^2x^2}h\,dx.$$

Multiplying this equation by
$$\int_0^\infty e^{-t^2}\,dt = \int_0^\infty e^{-h^2}\,dh,$$

we have
$$\left[\int_0^\infty e^{-t^2}\,dt\right]^2 = \int_0^\infty\int_0^\infty e^{-h^2(1+x^2)}\,h\,dx\,dh$$
$$= \int_0^\infty -\frac{1}{2(1+x^2)}\,dx\int_0^\infty e^{-h^2(1+x^2)}(-2h)(1+x^2)\,dh$$
$$= \frac{1}{2}\int_0^\infty \frac{dx}{1+x^2} = \frac{1}{2}\Big[\tan^{-1}x\Big]_0^\infty = \frac{\pi}{4}.$$

Therefore
$$\int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$$

and
$$\frac{\sqrt{\pi}}{2} = \frac{h}{2k},$$

or
$$k = \frac{h}{\sqrt{\pi}}, \qquad [178]$$

which shows the relation between the two constants.

The equation of the curve of error may now be written
$$y = \frac{h}{\sqrt{\pi}}e^{-h^2x^2}. \qquad [179]$$

214. The Average Error.

The average error (η) is the arithmetical mean of the errors, all taken with the same sign. To derive an expression for the average error, we see from equation (142) that $f(x)\,dx$ is the probability that an observation will fall between the limits x and $x + dx$; that is, it represents the proportion of all the errors that will probably fall within these limits. Hence, if n observations are made, the number in this strip will be $n\,f(x)\,dx$. The

sum of all the observations will be

$$n\int_{-\infty}^{\infty} xf(x)\,dx,$$

or

$$2\,n\int_{0}^{\infty} xf(x)\,dx.$$

The average error equals the sum of the errors divided by the number, that is,

$$\begin{aligned}\eta &= 2\int_{0}^{\infty} xf(x)\,dx\\ &= \frac{2\,h}{\sqrt{\pi}}\int_{0}^{\infty} e^{-h^2x^2}x\,dx\\ &= -\frac{1}{h\sqrt{\pi}}\int_{0}^{\infty} e^{-h^2x^2}(-2\,h^2x)\,dx\end{aligned}$$

$$h\sqrt{\pi} \qquad [180]$$

215. The Mean Square Error.

The mean square error (μ) of an observation is the square root of the arithmetical mean of the squares of the errors. Since the number of errors between x and $x + dx$ is $nf(x)\,dx$, the sum of the squares of these errors is

$$nx^2f(x)\,dx.$$

The sum of the squares of all the errors is

$$n\int_{-\infty}^{\infty} x^2f(x)\,dx.$$

Therefore

$$\mu^2 = \frac{h}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-h^2x^2}x^2\,dx. \qquad (d)$$

But

$$\frac{h}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-h^2x^2}\,dx = 1, \quad \text{or} \quad \int_{-\infty}^{\infty} e^{-h^2x^2}\,dx = \frac{\sqrt{\pi}}{h}.$$

If we differentiate this with respect to h as the independent variable, we obtain

$$-2\,h\int_{-\infty}^{\infty} e^{-h^2x^2}x^2\,dx = -\frac{\sqrt{\pi}}{h^2}. \qquad (e)$$

Substituting (*e*) in (*d*),

$$\mu \qquad h\sqrt{2} \tag{181}$$

216. The Probable Error.

The probable error (r) of an observation is an error such that one half the errors of the series are greater than it and the other half are less than it; that is, the probability of making an error greater than r is just equal to the probability of making an error less than r.

The probability that an error of an observation will fall between the limits x and $x + dx$ is $f\,(x)\,dx$. The probability that the error will fall between the limits $+r$ and $-r$ is given by

$$P = \int_{-r}^{+r} f(x)\,dx = \frac{2\,h}{\sqrt{\pi}} \int_0^{+r} e^{-h^2x^2}\,dx = \tfrac{1}{2},$$

by the definition.

To integrate, let $t = hx$, and $dt = h\,dx$,

then $$\frac{2}{\sqrt{\pi}} \int_0^{hr} e^{-t^2}\,dt = \tfrac{1}{2}.$$

If we evaluate this integral for assumed values of hr and then interpolate for the value of hr corresponding to $\frac{1}{2}$, we find it to be 0.47694.

Therefore $$r = \frac{0.47694}{h}. \tag{182}$$

All the precision measures have now been expressed in terms of h, and it is evident that,

$$r = 0.8453\,\eta \tag{183}$$
$$= 0.6745\,\mu. \tag{184}$$

The mean square error (μ) is the largest, and the probable error (r) is the smallest, of the three precision measures.

Any one of the three precision measures may be used to compare the relative accuracy of different series of observations, provided the different series are made under the same condi-

tions, so as to be affected by the same constant errors. In Europe the mean square error has been used more than the probable error; in the United States the probable error is generally employed. There are some advantages, however, in the use of the average error (η). Theoretically it is slightly less accurate than either of the others; but inasmuch as the quantity itself is an estimate of an uncertainty in measurement, this objection is not a serious one. The value of η lies between the values of μ and r. The method of computing η is simpler, as will be shown later, than the computation of either μ or r.

Since in Equa. [158] it was shown that p varies as h^2, it follows that

$$p \propto \frac{1}{\eta^2} \propto \frac{1}{\mu^2} \propto \frac{1}{r^2} \propto h^2; \tag{185}$$

that is, the weights of the different observations on a quantity vary inversely as the squares of the precision measures.

If μ is the precision measure of a direct observation of weight 1, and μ_0 is the precision measure of the mean, then since the weight of the mean is n, the number of observations,

$$\mu_0 = \frac{\mu}{\sqrt{n}}. \tag{f}$$

217. Computation of the Precision Measures.

Direct Observations of Equal Weight. To find μ, the mean square error of an observation, suppose that we have n direct observations of equal weight made on a quantity M, and that the results are M_1, M_2, . . . , and that M_0 is the most probable value. Let the errors be x_1, x_2, . . . and the residuals v_1, v_2, ...

Then in this case the residuals are

$$v_1 = M_1 - M_0,$$
$$v_2 = M_2 - M_0$$

and

$$\mu = \sqrt{\frac{\sum x^2}{n}}$$

If M_0 were the true value of M, the residuals would be the same as the true errors, and in that case

$$\mu = \qquad [186]$$

But in any limited number of observations this is not sufficiently exact. To obtain a more accurate expression, place

$$M_0 + x_0 = M;$$

then

$$x_1 = M_1 - (M_0 + x_0) = v_1 - x_0,$$
$$x_2 = M_2 - (M_0 + x_0) = v_2 - x_0,$$

Squaring, adding, and dividing by n,

$$\frac{\sum x^2}{n} = \mu^2 = \frac{1}{n}\left(\sum v^2 - 2\,x_0 \sum v + n x_0^2\right).$$

Since $\sum v = 0$, Art. 195, this reduces to

$$\frac{\sum x^2}{} = \frac{\sum v^2}{} + x_0^2.$$

The real value of x_0 is unknown; it may be taken as approximately equal to the mean square error of M_0, which, from Equa. (f), is

$$\mu_0 = \frac{\mu}{\sqrt{n}}; \qquad [187]$$

whence
$$\mu^2 = \frac{\sum v^2}{n} + \frac{\mu^2}{n}$$

Therefore
$$\mu = \sqrt{\frac{\sum v^2}{n - 1}}. \qquad [18$$

To find μ_0, the mean square error of the mean value, we have, by Equa. (f),

$$\mu_0 = \sqrt{\frac{\sum v^2}{n\,(n - 1)}}. \qquad [189]$$

From Equa. [184],

$$r = 0.6745 \sqrt{\frac{\sum v^2}{n-1}} \qquad [190]$$

and

$$r_0 = 0.6745 \sqrt{\frac{\sum v^2}{n(n-1)}}. \qquad [191]$$

To find the average error (η) of a single observation, we see that, from Equa. [188],

$$\sum v^2 = (n-1) \frac{\sum x^2}{n}.$$

On the average the values of these residuals will be

$$v_1 = \sqrt{\frac{n-1}{n}} \cdot x_1,$$

$$v_2 = \sqrt{\frac{n-1}{n}} \cdot x_2.$$

Adding and dividing by n,

$$\frac{\sum v}{n} = \sqrt{\frac{n-1}{n}} \cdot \frac{\sum x}{n} = \sqrt{\frac{n-1}{n}} \cdot \eta.$$

Therefore

$$\eta = \frac{\sum v}{\sqrt{n(n-1)}}, \qquad [192]$$

and

$$\eta_0 = \frac{\sum v}{n\sqrt{n-1}}. \qquad [193]$$

The probable error is sometimes computed from the average error in order to avoid computing the squares of the residuals. From Equa. [183],

$$r = \frac{0.8453 \sum v}{\sqrt{n(n-1)}}, \qquad [194]$$

and

$$r_0 = \frac{0.8453 \sum v}{n\sqrt{n-1}}. \qquad [195]$$

Evidently the mean error may also be computed from η.

218. Observations of Unequal Weights.

If the observations have unequal weights, let p_1, p_2, etc., be the weights; then

$$\mu_0 = \frac{\mu_1}{\sqrt{\sum p}}, \qquad \mu_k = \frac{\mu_1}{\sqrt{p_k}}, \quad \text{etc.}$$

By Art. 199, if each observation is multiplied by the square root of its weight, the observations are all reduced to weight unity. The residuals are therefore

$$v_1 \sqrt{p_1}, \qquad v_2 \sqrt{p_2}, \quad \text{etc.}$$

Applying Formulæ [188] to [195] to these residuals, we have

$$\mu_1 = \sqrt{\frac{\sum pv^2}{n - 1}}. \qquad [196]$$

$$\mu_k = \sqrt{\frac{\sum pv^2}{p_k (n - 1)}}. \qquad [197]$$

$$\mu_0 = \sqrt{\frac{\sum pv^2}{\sum p (n - 1)}}. \qquad [198]$$

$$r_1 = 0.6745 \sqrt{\frac{\sum pv^2}{n - 1}}. \qquad [199]$$

$$r_k = 0.6745 \sqrt{\frac{\sum pv^2}{p_k (n - 1)}}. \qquad [200]$$

$$r_0 = 0.6745 \sqrt{\frac{\sum pv^2}{\sum p (n - 1)}}. \qquad [201]$$

Also,

$$\eta_1 = \frac{\sum v \sqrt{p}}{\sqrt{n (n - 1)}}, \qquad [202]$$

$$\eta_k = \frac{\sum v \sqrt{p}}{\sqrt{p_k (n - 1)}} \qquad [203]$$

$$\eta_0 = \frac{\sum v\sqrt{p}}{\sqrt{\sum p\,(n-1)}}, \tag{204}$$

from which

$$r_1 = 0.8453\,\eta_1, \tag{205}$$

$$r_k = 0.8453\,\eta_k, \tag{206}$$

$$r_0 = 0.8453\,\eta_0. \tag{207}$$

219. Precision of Functions of the Observed Quantities. Suppose that a quantity M is defined by

$$M = M_1 + M_2,$$

where M_1 and M_2 are independent and are observed directly. Let the mean square error (m.s.e.) of M_1 be μ_1, and let that of M_2 be μ_2, the m.s.e. of the function M being denoted by μ_F. If we suppose the errors in the determination of M_1 to be x_1', x_1'', x_1''', . . . , and those of M_2 to be x_2', x_2'', x_2''', . . . , then the real errors of M, computed from the separate observations on M_1 and M_2, will be

$$x_1' \pm x_2', \quad x_1'' \pm x_2'', \cdots,$$

and
$$\mu_F^2 = \frac{(x_1' \pm x_2')^2 + (x_1'' + x_2'')^2 + \cdots}{n}$$

$$= \frac{\sum x_1^2 + 2\sum x_1x_2 + \sum x_2^2}{n}.$$

But the $\sum x_1x_2$ terms will cancel out, because in the long run there will be as many + as − products x_1x_2 of the same magnitude.

Therefore
$$\mu_F^2 = \mu_1^2 + \mu_2^2. \tag{208}$$

From Equas. [183] and [184] it is evident that

$$r_F^2 = r_1^2 + r_2^2 \tag{209}$$

and
$$\eta_F^2 = \eta_1^2 + \eta_2^2. \tag{210}$$

Let us suppose that the function is defined by

$$M = a_1M_1,$$

where a_1 is a constant; then the real errors of M will be

$$a_1x_1', \quad a_1x_1'', \quad a_1x_1''', \ldots,$$

and
$$\mu_F^2 = \frac{a_1^2\sum x_1^2}{n} = a_1^2\mu_1^2,$$

or
$$\mu_F = a_1\mu_1. \tag{211}$$

By combining [208] with [211] it is clear that if

$$M = a_1M_1 + a_2M_2 + a_3M_3 + \cdots,$$

then
$$\mu_F^2 = \sum a^2\mu^2, \tag{212}$$
$$r_F^2 = \sum a^2r^2, \tag{213}$$
$$\eta_F^2 = \sum a^2\eta^2. \tag{214}$$

Suppose that the function is of the general form indicated by

$$M = f(M_1, M_2, M_3, \ldots). \tag{g}$$

Let $M_1 = a_1 + m_1$, $M_2 = a_2 + m_2$, etc., in which a_1 is a close approximation to M_1, a_2 is a close approximation to M_2, and m_1 and m_2 are small corrections such that their squares may be neglected. We may regard m_1 and m_2, etc., as containing the real errors of $M_1, M_2, \ldots$, and $\mu_1, \mu_2, \ldots$ may be considered as the mean square errors of m_1, m_2, etc. Substituting in (g), we have

$$M = f((a_1 + m_1), (a_2 + m_2) \ldots).$$

Expanding this function by Taylor's theorem and denoting $f(a_1, a_2, \ldots)$ by M',

$$M = M' + m_1\frac{\partial M'}{\partial a_1} + m_2\frac{\partial M'}{\partial a_2} + \cdots, \tag{h}$$

in which the terms containing the squares and higher powers of $m_1, m_2, \ldots$ have been omitted. Then the m.s e. of M is the same as the m.s.e. of the terms in (h).

By Equa. [212], this is

$$\mu_F^2 = \mu_1^2\left[\frac{\partial M'}{\partial a_1}\right]^2 + \mu_2^2\left[\frac{\partial M'}{\partial a_2}\right]^2 +$$

or, with sufficient accuracy,

$$\mu_F^2 = \mu_1^2\left[\frac{\partial M}{\partial M_1}\right]^2 + \mu_2^2\left[\frac{\partial M}{\partial M_2}\right]^2 + \cdots. \qquad [215]$$

Similarly,

$$r_F^2 = r_1^2\left[\frac{\partial M}{\partial M_1}\right]^2 + r_2^2\left[\frac{\partial M}{\partial M_2}\right]^2 + \cdots, \qquad [216]$$

and

$$\eta_F^2 = \eta_1^2\left[\frac{\partial M}{\partial M_1}\right]^2 + \eta_2^2\left[\frac{\partial M}{\partial M_2}\right]^2 + \cdots. \qquad [217]$$

It should be observed that in the preceding cases the unknowns are supposed to be independent of each other. If the quantities M_1, M_2, etc., are functions of the same variable, a different procedure is necessary.

Also, in case the unknowns are subject to any number of conditions, the computation of the precision measure of any function must be so modified as to take into account the effect of these conditions.

220. Indirect Observations.

The computation of the precision of the adjusted values in the case of indirect observations is more complicated than in the case of direct observations, because it is necessary to know the weight of each of the unknowns, and this can only be found by the solution of equations similar to the normal equations

It may be shown that if there are n observations on q unknowns, then

$$\sqrt{\frac{\sum v^2}{n-q}}, \qquad [218]$$

where μ is the m s.e. of an observation of weight unity.

If p_z is the weight of an unknown, then the m.s.e. of this unknown is

$$\mu_z = \frac{\mu}{\sqrt{p_z}} = \sqrt{\frac{\sum v^2}{p_z(n-q)}}. \qquad [219]$$

Similarly,
$$r = 0.6745\sqrt{\frac{\sum v^2}{n-q}}, \qquad [220]$$

$$r_z = 0.6745\sqrt{\frac{\sum v^2}{p_z(n-q)}}, \qquad [221]$$

and
$$\eta = \frac{\sum v}{\sqrt{n(n-q)}}, \qquad [222]$$

$$\eta_z = \frac{\sum v}{\sqrt{p_z n(n-q)}}. \qquad [223]$$

221. Caution in the Application of Least Squares.

In applying the preceding principles it should be kept in mind that the ordinary adjustment by the method of least squares deals with the accidental errors only and can tell us nothing about the constant or systematic errors which may affect the results of observation. The "probable error" may therefore be far from the true error because such constant errors are present. We should think of the precision measures as indicating the deviation of the result from the mean result of a large number of such observations, rather than its deviation from the true value. It is usually true that the constant or the systematic errors are far more serious than the accidental errors, the observer should be continually on the watch for constant errors which may affect his result. So long as the conditions under which a measurement is made remain exactly the same the systematic errors are likely to be the same and are therefore not observed. The presence of such errors is most likely to be observed when the conditions are varied as much as possible. If observations are made at different temperatures, or under different conditions of illumination, or with different instruments, the variations of the results are usually greater than when the conditions are not changed. These variations indicate the presence of systematic errors and often enable the observer to estimate their magnitude.

The computation of the most probable value improves the

result with respect to the accidental errors, but leaves the more serious form of error untouched. The futility of multiplying observations and adjusting them for the purpose of removing the small accidental errors, and at the same time failing to remove the large constant error, may be illustrated by the results obtained by a marksman who holds his rifle steadily and places all his shots in a small group, but whose rifle sights are so far out of alignment that his shots all strike far from the bull's-eye. Of what use is the large number of shots under those circumstances? An adjustment of his results by least squares would correspond to an attempt to find the center of his group of shots, and would tell nothing about the distance from the bull's-eye. A study of the causes of the error so that he could make an adjustment of his sights would accomplish more toward hitting the mark than an infinite number of shots fired under the original conditions. Of course the comparison is quite untrue in one respect; the marksman knows where his mark is, while the observer can never know the true value of the quantity he is measuring.

While the method of least squares may not show directly the presence of constant errors, a study of the precision of the results, and a knowledge of the law governing the behavior of accidental errors, may enable the observer to detect the presence of constant error, or at least to decide whether it is probably present, and consequently to modify his methods of observing so as to reduce the effect of such constant error. Variations in the result which are greater than the error of observation shown by the precision measures is likely to mean that systematic error is present. This tracing of errors to their sources, and the consequent modification of instruments and methods, may constitute the most important application of least squares.

REFERENCES

Following are a few references to extended works on the subject of Least Squares.

BARTLETT, The Method of Least Squares (an Introductory Treatise).
CHAUVENET, Treatise on the Method of Least Squares. (Theory — Applications to Astronomy.)
CRANDALL, Geodesy and Least Squares. (Applications to Geodesy.)
MERRIMAN, Treatise on the Method of Least Squares.
UNITED STATES COAST AND GEODETIC SURVEY, Special Publications Nos. 28 and 138, and Serial No. 250. (Practice of the United States Coast and Geodetic Survey.)
WRIGHT AND HAYFORD, Adjustment of Observations. (Applications to Geodesy.)
LELAND, Practical Least Squares.

PROBLEMS

Problem 1. The following angles are measured at station O.

AOB =	31°	10′	15″.6	weight	(1)
BOC =	19	21	17 .4	"	(1)
AOC =	50	31	33 .5	"	(2)
COD =	38	50	16 .0	"	(2)
BOD =	58	11	32 .0	"	(1)
AOD =	89	21	51 .5	"	(1)

Adjust the angles

Problem 2. The angles of a triangle are as follows:

A	53° 53′ 38″.94	wt.	(3)
B	79 22 56 .17	"	(4)
C	46 43 29 .27	"	(2)

The spherical excess is 2″.83.
Adjust the triangle.

Problem 3. The angles of a quadrilateral are as follows, the numbers corresponding to those in Fig. 153. The weights are all unity. The spherical excess may be neglected.

1.	23°	31′	12″.5
2.	37	01	22 .5
3.	67	35	38 .3
4.	51	51	26 .7
5.	29	56	50 .0
6.	30	35	33 .2
7.	72	37	35 .0
8.	46	49	47 .5

The sum angles are

8 + 1	70° 21′ 05″.0
2 + 3	104 37 00 .0
4 + 5	81 48 20 .8
6 + 7	103 13 08 .4

Adjust the quadrilateral

Problem 4. Adjust the quadrilateral given on page 372, taking the pole at *O*, Fig. 153.

FORMULÆ AND TABLES

FORMULÆ

SERIES

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \cdots .$$

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots .$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots .$$

$$\sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \cdots .$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}.$$

BINOMIAL THEOREM

$$(a+b)^m = a^m + ma^{m-1}b + \frac{m(m-1)}{\lfloor 2} a^{m-2}b^2 + \cdots .$$

MACLAURIN'S THEOREM

$$f(x) = f(o) + \frac{x}{1} f'(o) + \frac{x^2}{\lfloor 2} f''(o) + \frac{x^3}{\lfloor 3} f'''(o) + \cdots .$$

TAYLOR'S THEOREM

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{\lfloor 2} f''(x) + \frac{h^3}{\lfloor 3} f'''(x) + \cdots$$

LOGARITHMIC SERIES

$$\log(1+x) = M\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots .\right)$$

$$\log(1-x) = -M\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots .\right)$$

OTHER SERIES

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots .$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots .$$

ELLIPSE AND SPHEROID

$$e^2 = \frac{a^2 - b^2}{}.$$

$$f = \frac{a - b}{a}.$$

$$R_m = \frac{a\,(1 - e^2)}{(1 - e^2 \sin^2 \phi)^{\frac{3}{2}}}.$$

$$N = \frac{a}{(1 - e^2 \sin^2 \phi)^{\frac{1}{2}}}.$$

$$R_\alpha = \frac{N R_m}{N \cos^2 \alpha + R_m \sin^2}$$

$$\text{Mean radius} = \rho = \sqrt{N R_m}.$$

CONSTANTS

$$\log_{10} x = M \log_e x.$$

M = modulus of system of common logarithms

= 0.434 2945.

$\log M$ = 9.637 7843.

π = 3.141 592 65. log = 0.497 1499.

$$\frac{180}{\pi} = 57.29577. \qquad \log = 1.758\ 1226.$$

$$\frac{180^\circ \times 60'}{\pi} = 3437.747. \qquad \log = 3.536\ 2739$$

$$\frac{180^\circ \times 60' \times 60''}{-} = 206\ 264.8. \qquad \log = 5.314\ 4251.$$

$$\text{arc } 1'' \quad \sin 1'' \quad \frac{1}{\tan 1''}. \quad \text{(Approx.)}$$

arc 1″ = 0.000 004 848 137. log = 4.685 5749.

$$\frac{1}{\text{arc } 1''} = 206\ 264.806 = \text{number of seconds in the radian.}$$

arc 1″ = about 0.3 inch at distance of one mile.

CLARKE SPHEROID (1866)

a = 6 378 206.4 meters. log = 6.804 6986.
b = 6 356 583.8 meters. log = 6.803 2238.

e^2	.006 768 658	log 7.830 5026 − 10
$1 - e^2$		9.997 0504 − 10
$a\,(1 - e^2)$		6.801 7490
$\dfrac{e^2 \sin^2 1''}{6\,(1 - e^2)}$		6.426 4506 − 20
$\dfrac{e^2 \sin 1''}{4\,(1 - e^2)}$		1.916 9670 − 10
$\dfrac{\sin^2 1''}{12}$		8.291 9685 − 20
$\sin 1''$		4.685 5749 − 10

COAST SURVEY SPHEROID (1909)

$$a = 6\ 378\ 388 \pm 18 \text{ meters.}$$

$$\frac{1}{f} = 297.0 \pm 0.5.$$

$$b = 6\ 356\ 909 \text{ meters.}$$

RELATION BETWEEN UNITS OF LENGTH

(Legal) Meters in one foot	= 0.304 8006.	log = 9.484 0158.
Feet in one (legal) meter	= 3.280 8333.	log = 0.515 9842.
Inches in one (legal) meter	= 39.37.	

TABLE I.—TABLE FOR DETERMINING RELATIVE STRENGTH OF FIGURES IN TRIANGULATION

°	10°	12	14°	16°	18°	20°	22°	24°	26°	28°	30°	35°	40°	45°	50°	55°	60°	65°	70°	75°	80°	85°	90°
10	428	359																					
12	359	295	253																				
14	315	253	214	187																			
16	284	225	187	162	143																		
18	262	204	168	143	126	113																	
20	245	189	153	130	113	100	91																
22	232	177	142	119	103	91	81	74															
24	221	167	134	111	95	83	74	67	61														
26	213	160	126	104	89	77	68	61	56	51													
28	206	153	120	99	83	72	63	57	51	47	43												
30	199	148	115	94	79	68	59	53	48	43	40	33											
35	188	137	106	85	71	60	52	46	41	37	33	27	23										
40	179	129	99	79	65	54	47	41	36	32	29	23	19	16									
45	172	124	93	74	60	50	43	37	32	28	25	20	16	13	11								
50	167	119	89	70	57	47	39	34	29	26	23	18	14	11	[illegible]	8							
55	162	115	86	67	51	44	37	32	27	24	21	16	12	10	8	7	5						
60	159	112	83	64	51	42	35	30	25	22	19	14	11	9	7	5	4	4					
65	155	109	80	62	49	40	33	28	24	21	18	13	10	7	6	5	4	3	2				
70	152	106	78	60	48	38	32	27	23	19	17	12	9	7	5	4	3	2	2	1			
75	150	104	76	58	46	37	30	25	21	18	16	11	8	6	4	3	2	2	1	1	1		
80	147	102	74	57	45	36	29	24	20	17	15	10	7	5	4	3	2	1	1	1	0	0	
85	145	100	73	55	43	34	28	23	19	16	14	10	7	5	3	2	2	1	1	0	0	0	0
90	143	98	71	54	42	33	27	22	19	16	13	9	6	4	3	2	1	1	1	0	0	0	0
95	140	96	70	53	41	32	26	22	18	15	13	9	6	4	3	2	1	1	0	0	0	0	
100	138	95	68	51	40	31	25	21	17	14	12	8	6	4	3	2	1	1	0	0	0		
105	136	93	67	50	39	30	25	20	17	14	12	8	5	4	2	2	1	1	0	0			
110	134	91	65	49	38	30	24	19	16	13	11	7	5	3	2	2	1	1	1				
115	132	89	64	48	37	29	23	19	15	13	11	7	5	3	2	2	1	1					
120	129	88	62	46	36	28	22	18	15	12	10	7	5	3	2	2	1						
125	127	86	61	45	35	27	22	18	14	12	10	7	5	4	3	2							
130	125	84	59	41	34	26	21	17	14	12	10	7	5	4	3								
135	122	82	58	43	33	26	21	17	14	12	10	7	5	4									
140	119	80	56	42	32	25	20	17	14	12	10	8	6										
145	116	77	55	41	32	25	21	17	15	13	11	9											
150	112	75	54	40	32	26	21	18	16	15	13												
152	111	75	53	40	32	26	22	19	17	16													
154	110	74	53	41	33	27	23	21	19														
156	108	74	54	42	34	28	25	22															
158	107	74	54	43	35	30	27																
160	107	74	56	45	38	33																	
162	107	76	59	48	42																		
164	109	79	63	54																			
166	113	86	71																				
168	122	98																					
170	143																						

TABLE II. — CORRECTION FOR EARTH'S CURVATURE AND REFRACTION

Dist.	Corr.	Dist.	Corr.	Dist.	Corr.
Miles.	Feet.	Miles.	Feet.	Miles.	Feet.
1	0.6	21	253 1	41	964.7
2	2.3	22	277.7	42	1012.2
3	5.2	23	303.6	43	1061.0
4	9.2	24	330 5	44	1111.0
5	14 4	25	358.6	45	1162.0
6	20.6	26	388.0	46	1214.2
7	28.1	27	418.3	47	1267.7
8	36 7	28	449 9	48	1322.1
9	46 4	29	482.6	49	1377.7
10	57.4	30	516.4	50	1434.6
11	69.4	31	551.4	51	1492.5
12	82.7	32	587.6	52	1551.6
13	97.0	33	624.9	53	1611.9
14	112.5	34	663.3	54	1673.3
15	129.1	35	703.0	55	1735.8
16	146 9	36	743.7	56	1799.6
17	165.8	37	785.6	57	1864.4
18	185 9	38	828.6	58	1930.4
19	207.2	39	872.8	59	1997.5
20	229.5	40	918.1	60	2065.8

TABLE III. — SHORT TABLE OF FACTORS FOR REDUCTION OF TRANSIT OBSERVATIONS

Top Argument = Star's Declination (δ).
Side Argument = Star's Zenith Distance (ζ).

[For factor A use left-hand argument. For factor B use right-hand argument. For factor C use bottom line.]

ζ	0°	10°	15°	20°	25°	30°	35°	40°	45°	50°	55°	60°	65°	70°	ζ
1°	0 02	0 02	0 02	0 02	0 02	0 02	0 02	0 02	0 02	0 03	0.03	0 03	0 04	0 05	89°
5	0.09	0 09	0 09	0.09	0.10	0 10	0 11	0 11	0 12	0 13	0 15	0 17	0 21	0 25	85
10	0 17	0 18	0 18	0 19	0.19	0 20	0.21	0 23	0 25	0 27	0 30	0 35	0 41	0.51	80
15	0 26	0.26	0 27	0 28	0 29	0 30	0 32	0 34	0 37	0 40	0 45	0 52	0 61	0 76	75
20	0 34	0 35	0 35	0.36	0 38	0 40	0 42	0 45	0 48	0 53	0 60	0 68	0 81	1 00	70
25	0 42	0 43	0 44	0 45	0 47	0 49	0 52	0 55	0 60	0 66	0 74	0 85	1 00	1 24	65
30	0.50	0.51	0.52	0 53	0 55	0 58	0 61	0 65	0 71	0 78	0 87	1 00	1 18	1 46	60
35	0 57	0 58	0.59	0 61	0 63	0 66	0 70	0 75	0 81	0 89	1 00	1 15	1 36	1 68	55
40	0 64	0 65	0 67	0 68	0 71	0 74	0 78	0 84	0 91	1 00	1 12	1 29	1 52	1 88	50
45	0 71	0 72	0 73	0 75	0 78	0 82	0 86	0 92	1 00	1 10	1 23	1 41	1 67	2 07	45
50	0 77	0 78	0 79	0 82	0 85	0.89	0 94	1 00	1 08	1 19	1.34	1.53	1 81	2.24	40
55	0 82	0.83	0 85	0 87	0.90	0 95	1 00	1 07	1.16	1 27	1 43	1 64	1.94	2.40	35
60	0 87	0 88	0 90	0 92	0.96	1.00	1.06	1 13	1.22	1.35	1.51	1.73	2 05	2.53	30
65	0 91	0.92	0 94	0 96	1.00	1.05	1.11	1 18	1 28	1.41	1.58	1 81	2 14	2.65	25
70	0 94	0.95	0 97	1 00	1.04	1.09	1.15	1 23	1 33	1 46	1.64	1.88	2 22	2.75	20
75	0 97	0 98	1 00	1 03	1.07	1 12	1 18	1 26	1 37	1 50	1 68	1.93	2 29	2 82	15
80	0 98	1 00	1 02	1.05	1 09	1 14	1 20	1 29	1 39	1 53	1 72	1 97	2 33	2.88	10
85	1 00	1 01	1 03	1 06	1 10	1 15	1 22	1 30	1 41	1 55	1 74	1 99	2 36	2.91	5
90	1 00	1 02	1 04	1 06	1 10	1 15	1 22	1 31	1 41	1 56	1 74	2 00	2 37	2 92	0

TABLE IV. — DIURNAL ABERRATION (κ)

Latitude = ϕ.	Declination = δ.										
	0°	10°	20°	30°	40°	50°	60°	70°	75°	80°	85°
°	s	s	s	s	s	s	s	s	s	s	s
0	0 02	0 02	0 02	0 02	0 03	0 03	0 04	0 06	0 08	0.12	0 24
10	0 02	0 02	0 02	0 02	0 03	0 03	0 04	0 06	0 08	0 12	0.24
20	0.02	0.02	0 02	0 02	0 03	0 03	0 04	0 06	0 08	0 11	0 23
30	0 02	0 02	0 02	0.02	0 02	0 03	0 04	0 05	0.07	0 10	0 21
40	0.02	0 02	0 02	0 02	0 02	0 03	0 03	0 05	0 06	0 09	0 18
50	0 01	0.01	0 01	0 02	0 02	0 02	0 03	0.04	0 05	0.08	0 15
60	0 01	0.01	0 01	0 01	0 01	0 02	0 02	0.03	0.04	0.06	0.12
70	0.01	0 01	0 01	0 01	0 01	0 01	0 01	0 02	0.03	0 04	0 08
80	0.00	0.00	0 00	0 00	0 00	0 01	0.01	0 01	0.01	0 02	0 04

TABLE V. — CORRECTION TO LATITUDE FOR DIFFERENTIAL REFRACTION = $\frac{1}{2}(r - r')$.

[The sign of the correction is the same as that of the micrometer difference.]

One-half diff. of zenith distances.	Zenith distance.							
	0°	10°	20°	25°	30°	35°	40°	45°
′	″	″	″	″	″	″	″	″
0.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.5	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02
1.0	0.02	0.02	0.02	0.02	0.02	0.03	0.03	0.03
1.5	0.03	0.03	0.03	0.03	0.03	0.04	0.04	0.05
2.0	0.03	0.03	0.04	0.04	0.04	0.05	0.06	0.07
2.5	0.04	0.04	0.05	0.05	0.06	0.06	0.07	0.08
3.0	0.05	0.05	0.06	0.06	0.07	0.08	0.09	0.10
3.5	0.06	0.06	0.07	0.07	0.08	0.09	0.10	0.12
4.0	0.07	0.07	0.08	0.08	0.09	0.10	0.11	0.13
4.5	0.08	0.08	0.09	0.09	0.10	0.11	0.13	0.15
5.0	0.08	0.09	0.10	0.10	0.11	0.13	0.14	0.17
5.5	0.09	0.10	0.10	0.11	0.12	0.14	0.16	0.18
6.0	0.10	0.10	0.11	0.12	0.13	0.15	0.17	0.20
6.5	0.11	0.11	0.12	0.13	0.14	0.16	0.19	0.22
7.0	0.12	0.12	0.13	0.14	0.16	0.18	0.20	0.23
7.5	0.13	0.13	0.14	0.15	0.17	0.19	0.21	0.25
8.0	0.13	0.14	0.15	0.16	0.18	0.20	0.23	0.27
8.5	0.14	0.15	0.16	0.17	0.19	0.21	0.24	0.29
9.0	0.15	0.16	0.17	0.18	0.20	0.23	0.26	0.30
9.5	0.16	0.16	0.18	0.19	0.21	0.24	0.27	0.32
10.0	0.17	0.17	0.19	0.20	0.22	0.25	0.29	0.34
10.5	0.18	0.18	0.20	0.21	0.23	0.26	0.30	0.35
11.0	0.18	0.19	0.21	0.22	0.25	0.28	0.31	0.37
11.5	0.19	0.20	0.22	0.23	0.26	0.29	0.33	0.39
12.0	0.20	0.21	0.23	0.25	0.27	0.30	0.34	0.40
12.5	0.21	0.22	0.24	0.26	0.28	0.31	0.36	0.42
13.0	0.22	0.22	0.25	0.27	0.29	0.33	0.37	0.44
13.5	0.23	0.23	0.26	0.28	0.30	0.34	0.39	0.45
14.0	0.23	0.24	0.27	0.29	0.31	0.35	0.40	0.47
14.5	0.24	0.25	0.28	0.30	0.32	0.36	0.41	0.49
15.0	0.25	0.26	0.29	0.31	0.34	0.38	0.43	0.50
15.5	0.26	0.27	0.29	0.32	0.35	0.39	0.44	0.52
16.0	0.27	0.28	0.30	0.33	0.36	0.40	0.46	0.54
16.5	0.28	0.29	0.31	0.34	0.37	0.41	0.47	0.55
17.0	0.29	0.29	0.32	0.35	0.38	0.43	0.49	0.57
17.5	0.29	0.30	0.33	0.36	0.39	0.44	0.50	0.59
18.0	0.30	0.31	0.34	0.37	0.40	0.45	0.51	0.60
18.5	0.31	0.32	0.35	0.38	0.41	0.46	0.53	0.62
19.0	0.32	0.33	0.36	0.39	0.43	0.48	0.54	0.64
19.5	0.33	0.34	0.37	0.40	0.44	0.49	0.56	0.65
20.0	0.34	0.35	0.38	0.41	0.45	0.50	0.57	0.67

TABLE VI. — CORRECTION TO LATITUDE FOR REDUCTION TO MERIDIAN

[Star off the meridian but instrument in the meridian. The sign of the correction to the latitude is positive except for stars south of the equator and subpolars.]

δ	10^s	15^s	20^s	22^s	24^s	26^s	28^s	30^s	32^s	34^s	36^s	38^s	δ
°	″	″	″	″	″	″	″	″	″	″	″	″	°
1										0 01	0 01	0 01	89
2					0 01	0 01	0 01	0 01	0 01	0 01	0 01	0 01	88
3			0 01	0 01	0 01	0 01	0 01	0 01	0 01	0 02	0 02	0 02	87
4			0 01	0 01	0 01	0 01	0 01	0 02	0 02	0 02	0 02	0 03	86
5		0 01	0 01	0 01	0 01	0 02	0 02	0 02	0 02	0 03	0 03	0 03	85
6		0 01	0 01	0 01	0 02	0 02	0 02	0 03	0 03	0 03	0 04	0 04	84
7		0 01	0 01	0 02	0 02	0 02	0 03	0 03	0 03	0 04	0 04	0 05	83
8		0 01	0 02	0 02	0 02	0 03	0 03	0 03	0 04	0 04	0 05	0 05	82
9		0 01	0 02	0 02	0 02	0 03	0 03	0 04	0 04	0 05	0 05	0 06	81
10		0 01	0 02	0 02	0 03	0 03	0 04	0 04	0 05	0 05	0 06	0 07	80
12	0 01	0 01	0 02	0 03	0 03	0 04	0 05	0 05	0 06	0 06	0 07	0 08	78
14	0 01	0 01	0 03	0 03	0 04	0 04	0 05	0 06	0 07	0 07	0 08	0 09	76
16	0 01	0 02	0 03	0 03	0 04	0 05	0 06	0 07	0 07	0 08	0 09	0 10	74
18	0 01	0 02	0 03	0 04	0 05	0 05	0 06	0 07	0 08	0 09	0 10	0 12	72
20	0 01	0 02	0 04	0 04	0 05	0 06	0 07	0 08	0 09	0 10	0 11	0 13	70
22	0 01	0 02	0 04	0 05	0 05	0 06	0 07	0 09	0 10	0 11	0 12	0 14	68
24	0 01	0 02	0 04	0 05	0 06	0 07	0 08	0 09	0 10	0 12	0 13	0 15	66
26	0 01	0 02	0 04	0 05	0 06	0 07	0 08	0 10	0 11	0 12	0 14	0 15	64
28	0 01	0 03	0 05	0 05	0 07	0 08	0 09	0 10	0 12	0 13	0 15	0 16	62
30	0 01	0 03	0 05	0 06	0 07	0 08	0 09	0 11	0 12	0 14	0 15	0 17	60
32	0 01	0 03	0 05	0 06	0 07	0 08	0 10	0.11	0 13	0 14	0 16	0 18	58
34	0 01	0 03	0 05	0 06	0 07	0 09	0 10	0 11	0 13	0 15	0 16	0 18	56
36	0 01	0 03	0 05	0 06	0 07	0 09	0 10	0 12	0 13	0 15	0 17	0 19	54
38	0 01	0 03	0 05	0 06	0 08	0 09	0 10	0 12	0 13	0 15	0 17	0 19	52
40	0 01	0 03	0 05	0 07	0 08	0 09	0 11	0 12	0 14	0 16	0 17	0 19	50
45	0 01	0 03	0 05	0 07	0 08	0 09	0 11	0 12	0 14	0 16	0 18	0 20	45

δ	40^s	42^s	44^s	46^s	48^s	50^s	52^s	54^s	56^s	58^s	60^s	δ
°	″	″	″	″	″	″	″	″	″	″	″	°
1	0 01	0 01	0 01	0 01	0 01	0 01	0 01	0 01	0 01	0 02	0 02	89
2	0 02	0 02	0 02	0.02	0 02	0 02	0 03	0 03	0 03	0 03	0 03	88
3	0 02	0 03	0 03	0 03	0 03	0 04	0 04	0 04	0 04	0 05	0 05	87
4	0 03	0 03	0 04	0 04	0 04	0 05	0 05	0 06	0 06	0 06	0 07	86
5	0 04	0 04	0 05	0 05	0 05	0 06	0 06	0 07	0 07	0 08	0 09	85
6	0 05	0 05	0 06	0 06	0 07	0 07	0 08	0 08	0 09	0 10	0 10	84
7	0 05	0 06	0 06	0 07	0 08	0 08	0 09	0 10	0 10	0 11	0 12	83
8	0 06	0 07	0 07	0 08	0 09	0 09	0 10	0 11	0 12	0 13	0 14	82
9	0.07	0 07	0 08	0 09	0 10	0 11	0 11	0 12	0 13	0 14	0 15	81
10	0 07	0 08	0.09	0 10	0 11	0 12	0 13	0 14	0 15	0 16	0 17	80
12	0 09	0 10	0 11	0 12	0 13	0 14	0 15	0 16	0 17	0 19	0 20	78
14	0 10	0 11	0 12	0 14	0 15	0 16	0 17	0 19	0 20	0 22	0 23	76
16	0 12	0 13	0 14	0 15	0 17	0 18	0 20	0 21	0 23	0 24	0 26	74
18	0 13	0 14	0 16	0 17	0 18	0 20	0 22	0 23	0 25	0 27	0 29	72
20	0.14	0 15	0 17	0 19	0 20	0 22	0 24	0 26	0 28	0 29	0 32	70
22	0 15	0 17	0 18	0 20	0 22	0 24	0 26	0 28	0 30	0 32	0 34	68
24	0.16	0 18	0 20	0 21	0 23	0 25	0 27	0 29	0 32	0 34	0 36	66
26	0 17	0 19	0 21	0 23	0 25	0 27	0 29	0 31	0 34	0 36	0 39	64
28	0 18	0 20	0.22	0 24	0 26	0 28	0 31	0 33	0 35	0 38	0 41	62
30	0 19	0 21	0 23	0 25	0 27	0 30	0 32	0 34	0 37	0 40	0 42	60
32	0 20	0 22	0 24	0 26	0 28	0 31	0 33	0 36	0 39	0 41	0.44	58
34	0 20	0 22	0 24	0 27	0 29	0 32	0 34	0 37	0 40	0 42	0 45	56
36	0 21	0 23	0 25	0 28	0 30	0 32	0 35	0 38	0 41	0 44	0 47	54
38	0 21	0 23	0 26	0 28	0 30	0 33	0 36	0 39	0 41	0 44	0 48	52
40	0 21	0 24	0 26	0 28	0 31	0 34	0 36	0 39	0 42	0 45	0 48	50
45	0 22	0 24	0.26	0 29	0 31	0 34	0 37	0 40	0 43	0 46	0 49	45

TABLE VII.—REDUCTION OF LATITUDE TO SEA LEVEL

[The correction is negative in every case.]

h \ ϕ		5° / 85°	10° / 80°	15° / 75°	20° / 70°	25° / 65°	30 / 60°	35° / 55°	40° / 50°	45°
Feet.	Meters.	″	″	″	″	″	″	″	″	″
100	30	0 00	0 00	0 00	0 00	0 00	0 00	0 00	0 01	0.01
200	61	0 00	0 00	0 01	0 01	0 01	0 01	0 01	0.01	0.01
300	91	0.00	0 01	0 01	0 01	0 01	0 01	0 01	0 02	0.02
400	122	0.00	0 01	0 01	0 01	0 02	0 02	0.02	0 02	0 02
500	152	0 00	0 01	0 01	0 02	0 02	0 02	0 02	0 03	0.03
600	183	0 01	0.01	0 02	0 02	0 02	0.03	0 03	0 03	0.03
700	213	0 01	0 01	0.02	0 02	0 03	0 03	0 03	0 04	0 04
800	244	0 01	0 01	0 02	0 03	0.03	0 04	0 04	0 04	0 04
900	274	0.01	0 02	0 02	0 03	0 04	0 04	0 04	0 05	0 05
1000	305	0 01	0 02	0 03	0 03	0 04	0.05	0 05	0 05	0 05
1100	335	0 01	0.02	0 03	0.04	0 04	0 05	0 05	0 06	0 06
1200	366	0.01	0.02	0 03	0 04	0 05	0 05	0 06	0 06	0 06
1300	396	0 01	0.02	0 03	0.04	0 05	0 06	0 06	0.07	0 07
1400	427	0 01	0.02	0 04	0 05	0 06	0 06	0 07	0.07	0 07
1500	457	0 01	0.03	0 04	0 05	0 06	0 07	0 07	0 08	0 08
1600	488	0 01	0 03	0 04	0 05	0 06	0 07	0 08	0.08	0 08
1700	518	0 02	0 03	0 04	0 06	0 07	0 08	0 08	0.09	0 09
1800	549	0 02	0 03	0 05	0 06	0 07	0 08	0 09	0 09	0 09
1900	579	0 02	0 03	0 05	0 06	0 08	0 09	0 09	0.10	0.10
2000	610	0 02	0 04	0 05	0 07	0 08	0 09	0 10	0.10	0 10
2100	640	0 02	0 04	0 05	0 07	0 08	0 09	0 10	0.11	0 11
2200	671	0 02	0 04	0 06	0 07	0 09	0 10	0 11	0.11	0.11
2300	701	0.02	0 04	0 06	0 08	0 09	0 10	0 11	0 12	0.12
2400	732	0.02	0 04	0 06	0 08	0 10	0 11	0 12	0.12	0.13
2500	762	0 02	0 04	0.07	0 08	0 10	0 11	0 12	0 13	0 13
2600	792	0 02	0.05	0.07	0 09	0 10	0 12	0.13	0.13	0.14
2700	823	0.02	0 05	0 07	0.09	0 11	0 12	0.13	0.14	0.14
2800	853	0.03	0 05	0 07	0.09	0 11	0.13	0.14	0.14	0 15
2900	884	0.03	0 05	0 08	0.10	0 12	0 13	0.14	0 15	0.15
3000	914	0 03	0.05	0 08	0 10	0 12	0 14	0 15	0.15	0.16
3100	945	0.03	0.06	0.08	0 10	0.12	0.14	0 15	0.16	0.16
3200	975	0.03	0.06	0.08	0.11	0 13	0.14	0.16	0.16	0.17
3300	1006	0 03	0 06	0 09	0.11	0 13	0.15	0 16	0.17	0.17
3400	1036	0.03	0.06	0.09	0.11	0.12	0.15	0 17	0.17	0.18
3500	1067	0 03	0 06	0.09	0 12	0 14	0 16	0.17	0.18	0.18

TABLE VII (*Con.*).—REDUCTION OF LATITUDE TO SEA LEVEL

[The correction is negative in every case]

h (φ)		5° / 85°	10° / 80°	15° / 75°	20° / 70°	25° / 65°	30° / 60°	35° / 55°	40° / 50°	45°
Feet.	Meters	″	″	″	″	″		″	″	″
3600	1097	0 03	0 06	0 09	0 12	0 14	0 16	0 18	0 18	0 19
3700	1128	0 03	0 07	0 10	0 12	0 15	0 17	0 18	0 19	0 19
3800	1158	0 03	0 07	0 10	0 13	0 15	0 17	0 19	0 20	0 20
3900	1189	0 04	0 07	0 10	0 13	0 16	0 18	0 19	0 20	0 20
4000	1219	0 04	0 07	0 10	0 13	0 16	0 18	0 20	0 21	0 21
4100	1250	0 04	0 07	0 11	0 14	0 16	0 19	0 20	0 21	0 21
4200	1280	0 04	0 07	0 11	0 14	0 17	0 19	0 21	0 22	0 22
4300	1311	0 04	0 08	0 11	0 14	0 17	0 19	0 21	0 22	0 22
4400	1341	0 04	0 08	0 11	0 15	0 18	0 20	0 22	0 23	0 23
4500	1372	0 04	0 08	0 12	0 15	0 18	0 20	0 22	0 23	0 23
4600	1402	0 04	0 08	0 12	0 15	0 18	0 21	0 23	0 24	0 24
4700	1433	0 04	0 08	0 12	0 16	0 19	0 21	0 23	0 24	0 24
4800	1463	0 04	0 09	0 13	0 16	0 19	0 22	0 24	0 25	0 25
4900	1494	0 04	0 09	0 13	0 16	0 20	0 22	0 24	0 25	0 26
5000	1524	0 05	0 09	0 13	0 17	0 20	0 23	0 24	0 26	0 26
5100	1554	0 05	0 09	0 13	0 17	0 20	0 23	0 25	0 26	0 27
5200	1585	0 05	0 09	0 14	0 17	0 21	0 23	0 25	0 27	0 27
5300	1615	0 05	0 09	0 14	0 18	0 21	0 24	0 26	0 27	0 28
5400	1646	0 05	0 10	0 14	0 18	0 22	0 24	0 26	0 28	0 28
5500	1676	0 05	0 10	0 14	0 18	0 22	0 25	0 27	0 28	0 29
5600	1707	0 05	0 10	0 15	0 19	0 22	0 25	0 27	0 29	0 29
5700	1737	0 05	0 10	0 15	0 19	0 23	0 26	0 28	0 29	0 30
5800	1768	0 05	0 10	0 15	0 19	0 23	0 26	0 28	0 30	0 30
5900	1798	0 05	0 11	0 15	0 20	0 24	0.27	0 29	0 30	0 31
6000	1829	0 05	0 11	0 16	0 20	0 24	0 27	0 29	0 31	0 31
6100	1859	0 06	0 11	0 16	0 20	0 24	0 28	0 30	0 31	0 32
6200	1890	0 06	0 11	0 16	0 21	0 25	0 28	0 30	0 32	0 32
6300	1920	0 06	0 11	0 16	0 21	0 25	0 28	0 31	0 32	0 33
6400	1951	0 06	0 11	0 17	0 21	0 26	0 29	0 31	0 33	0 33
6500	1981	0 06	0 12	0 17	0 22	0 26	0 29	0 32	0 33	0 34
6600	2012	0 06	0 12	0 17	0 22	0 26	0 30	0 32	0 34	0 34
6700	2042	0 06	0 12	0 17	0 22	0 27	0 30	0 33	0 34	0 35
6800	2073	0 06	0 12	0 18	0 23	0 27	0 31	0 33	0 35	0 35
6900	2103	0 06	0 12	0 18	0 23	0 28	0 31	0 34	0 35	0 36
7000	2134	0 06	0 12	0 18	0 23	0 28	0.32	0 34	0 36	0 36

TABLE VII (*Con.*).—REDUCTION OF LATITUDE TO SEA LEVEL

[The correction is negative in every case]

h	φ	5° / 85°	10° / 80°	15° / 75°	20° / 70°	25° / 65°	30° / 60°	35° / 55°	40° / 50°	45°
Feet.	Meters.	"	"	"	"	"	"	"	"	"
7100	2164	0 06	0 13	0 19	0 24	0.28	0 32	0 35	0 36	0.37
7200	2195	0 07	0 13	0 19	0 24	0 29	0.33	0.35	0 37	0 38
7300	2225	0.07	0 13	0 19	0 24	0 29	0 33	0 36	0 37	0 38
7400	2256	0 07	0 13	0 19	0 25	0 30	0 33	0 36	0 38	0 39
7500	2286	0 07	0 13	0 20	0 25	0 30	0 34	0 37	0 38	0 39
7600	2316	0 07	0 14	0 20	0 25	0 30	0 34	0 37	0 39	0.40
7700	2347	0 07	0 14	0 20	0 26	0 31	0 35	0 38	0 40	0 40
7800	2377	0 07	0 14	0 20	0 26	0 31	0 35	0 38	0 40	0 41
7900	2408	0 07	0 14	0 21	0 26	0 32	0 36	0 39	0 41	0.41
8000	2438	0 07	0 14	0 21	0 27	0 32	0 36	0 39	0 41	0 42
8100	2469	0 07	0 14	0 21	0.27	0 32	0 37	0.40	0 42	0 42
8200	2499	0 07	0.15	0 21	0.27	0 33	0 37	0 40	0 42	0 43
8300	2530	0 08	0 15	0 22	0 28	0 33	0 37	0 41	0 43	0 43
8400	2560	0 08	0 15	0 22	0 28	0 34	0 38	0 41	0 43	0 44
8500	2591	0 08	0 15	0 22	0 28	0 34	0 38	0 42	0 44	0 44
8600	2621	0 08	0 15	0 22	0 29	0 34	0 39	0 42	0 44	0 45
8700	2652	0 08	0 16	0 23	0 29	0 35	0 39	0 43	0 45	0 45
8800	2682	0 08	0.16	0 23	0 29	0 35	0 40	0 43	0 45	0 46
8900	2713	0 08	0 16	0 23	0 30	0 36	0 40	0 44	0 46	0 46
9000	2743	0 08	0 16	0 23	0 30	0 36	0 41	0 44	0 46	0 47
9100	2774	0.08	0 16	0.24	0 30	0 36	0 41	0 45	0 47	0 47
9200	2804	0 08	0 16	0 24	0 31	0 37	0 42	0 45	0 47	0 48
9300	2835	0 08	0 17	0 24	0 31	0 37	0 42	0 46	0 48	0 48
9400	2865	0 09	0 17	0 24	0 31	0 38	0 42	0 46	0 48	0 49
9500	2896	0 09	0 17	0 25	0 32	0 38	0 43	0 47	0 49	0 50
9600	2926	0 09	0 17	0 25	0 32	0 38	0 43	0 47	0 49	0 50
9700	2957	0 09	0 17	0 25	0 32	0 39	0 44	0 48	0 50	0 51
9800	2987	0 09	0 17	0 26	0 33	0 39	0 44	0 48	0 50	0 51
9900	3018	0 09	0 18	0.26	0 33	0 40	0 45	0 48	0.51	0 52
10000	3048	0 09	0 18	0 26	0 33	0 40	0 45	0 49	0 51	0 52

TABLE VIII.—FOR CONVERTING SIDEREAL INTO MEAN SOLAR TIME

[Increase in Sun's Right Ascension in Sidereal h. m. s.]

Mean Time = Sidereal Time − C'.

Sid. Hrs.	Corr.	Sid Min.	Corr.	Sid Min.	Corr	Sid Sec	Corr.	Sid. Sec	Corr.
	m s		s		s		s		s
1	0 9.830	1	0.164	31	5.070	1	0.003	31	0.085
2	0 19.659	2	0.328	32	5.242	2	0.005	32	0.087
3	0 29.489	3	0.491	33	5.406	3	0.008	33	0.090
4	0 39.318	4	0.655	34	5.570	4	0.011	34	0.093
5	0 49.148	5	0.819	35	5.734	5	0.014	35	0.096
6	0 58.977	6	0.983	36	5.898	6	0.016	36	0.098
7	1 8.807	7	1.147	37	6.062	7	0.019	37	0.101
8	1 18.636	8	1.311	38	6.225	8	0.022	38	0.104
9	1 28.466	9	1.474	39	6.389	9	0.025	39	0.106
10	1 38.296	10	1.638	40	6.553	10	0.027	40	0.109
11	1 48.125	11	1.802	41	6.717	11	0.030	41	0.112
12	1 57.955	12	1.966	42	6.881	12	0.033	42	0.115
13	2 7.784	13	2.130	43	7.045	13	0.035	43	0.117
14	2 17.614	14	2.294	44	7.208	14	0.038	44	0.120
15	2 27.443	15	2.457	45	7.372	15	0.041	45	0.123
16	2 37.273	16	2.621	46	7.536	16	0.044	46	0.126
17	2 47.102	17	2.785	47	7.700	17	0.046	47	0.128
18	2 56.932	18	2.949	48	7.864	18	0.049	48	0.131
19	3 6.762	19	3.113	49	8.027	19	0.052	49	0.134
20	3 16.591	20	3.277	50	8.191	20	0.055	50	0.137
21	3 26.421	21	3.440	51	8.355	21	0.057	51	0.139
22	3 36.250	22	3.604	52	8.519	22	0.060	52	0.142
23	3 46.080	23	3.768	53	8.683	23	0.063	53	0.145
24	3 55.909	24	3.932	54	8.847	24	0.066	54	0.147
		25	4.096	55	9.010	25	0.068	55	0.150
		26	4.259	56	9.174	26	0.071	56	0.153
		27	4.423	57	9.338	27	0.074	57	0.156
		28	4.587	58	9.502	28	0.076	58	0.158
		29	4.751	59	9.666	29	0.079	59	0.161
		30	4.915	60	9.830	30	0.082	60	0.164

TABLE IX.—FOR CONVERTING MEAN SOLAR INTO SIDEREAL TIME

[Increase in Sun's Right Ascension in Solar h. m. s.]

Sidereal Time = Mean Time + C.

Mean Hrs	Corr.	Mean Min	Corr	Mean Min	Corr	Mean Sec.	Corr.	Mean Sec	Corr.
	m s		s		s		s		s
1	0 9.856	1	0.164	31	5.093	1	0.003	31	0.085
2	0 19.713	2	0.329	32	5.257	2	0.005	32	0.088
3	0 29.569	3	0.493	33	5.421	3	0.008	33	0.090
4	0 39.426	4	0.657	34	5.585	4	0.011	34	0.093
5	0 49.282	5	0.821	35	5.750	5	0.014	35	0.096
6	0 59.139	6	0.986	36	5.914	6	0.016	36	0.099
7	1 8.995	7	1.150	37	6.078	7	0.019	37	0.101
8	1 18.852	8	1.314	38	6.242	8	0.022	38	0.104
9	1 28.708	9	1.478	39	6.407	9	0.025	39	0.107
10	1 38.565	10	1.643	40	6.571	10	0.027	40	0.110
11	1 48.421	11	1.807	41	6.735	11	0.030	41	0.112
12	1 58.278	12	1.971	42	6.900	12	0.033	42	0.115
13	2 8.134	13	2.136	43	7.064	13	0.036	43	0.118
14	2 17.991	14	2.300	44	7.228	14	0.038	44	0.120
15	2 27.847	15	2.464	45	7.392	15	0.041	45	0.123
16	2 37.704	16	2.628	46	7.557	16	0.044	46	0.126
17	2 47.560	17	2.793	47	7.721	17	0.047	47	0.129
18	2 57.417	18	2.957	48	7.885	18	0.049	48	0.131
19	3 7.273	19	3.121	49	8.049	19	0.052	49	0.134
20	3 17.129	20	3.285	50	8.214	20	0.055	50	0.137
21	3 26.986	21	3.450	51	8.378	21	0.057	51	0.140
22	3 36.842	22	3.614	52	8.542	22	0.060	52	0.142
23	3 46.699	23	3.778	53	8.707	23	0.063	53	0.145
24	3 56.555	24	3.943	54	8.871	24	0.066	54	0.148
		25	4.107	55	9.035	25	0.068	55	0.151
		26	4.271	56	9.199	26	0.071	56	0.153
		27	4.435	57	9.364	27	0.074	57	0.156
		28	4.600	58	9.528	28	0.077	58	0.160
		29	4.764	59	9.692	29	0.079	59	0.162
		30	4.928	60	9.856	30	0.082	60	0.164

TABLE X. — LENGTHS OF ARCS OF THE PARALLEL AND THE MERIDIAN AND LOGS OF N AND R_m

[Metric Units]

Latitude.		Parallel. Value of 1°	Meridian. Value of 1°.	Log N	Log R_m
°	′	Meters.	Meters.		
0	00	111,321	110,567 2	6 8046985	6.8017489
	30	1,361	567 3	6987	7493
1	00	1,304	567 6	6990	7502
	30	1,283	568 0	6996	7519
2	00	1,253	568 6	7003	7543
	30	1,215	569 4	7012	7573
3	00	1,169	570 3	7025	7610
	30	1,114	571 1	7040	7654
4	00	1,051	572 7	7057	7704
	30	110,980	574 1	7076	7761
5	00	110,900	110,575 8	6 8047097	6 8017824
	30	0,812	577 6	7120	7894
6	00	0,715	579 5	7146	7971
	30	0,610	581 6	7174	8054
7	00	0,497	583 9	7203	8144
	30	0,375	586 4	7235	8240
8	00	0,245	589 0	7270	8343
	30	0,106	591 8	7307	8452
9	00	109,959	594 7	7345	8568
	30	9,804	597 8	7385	8690
10	00	109,641	110,601 1	6 8047428	6 8018819
	30	9,469	604 5	7474	8954
11	00	9,289	608 1	7520	9094
	30	9,101	611 9	7570	9241
12	00	108,904	615 8	7620	9395
	30	8,699	619 8	7673	9555
13	00	8,486	624 1	7729	9720
	30	8,265	628 4	7786	9892
14	00	8,036	633 0	7845	6 8020070
	30	107,798	637 6	7907	0254
15	00	107,553	110,642 5	6 8047970	6 8020443
	30	7,299	647 5	8035	0639
16	00	7,036	652 6	8102	0839
	30	6,766	657 8	8171	1047
17	00	6,487	663 3	8242	1258
	30	6,201	668 8	8315	1477
18	00	5,906	674 5	8389	1701
	30	5,604	680 4	8465	1930
19	00	5,294	686 3	8544	2165
	30	4,975	692 4	8624	2404
20	00	104,649	110,698 7	6 8048705	6 8022649
	30	4,314	705 1	8789	2900
21	00	3,972	711.6	8874	3155
	30	3,622	718.2	8960	3415
22	00	3,264	725 0	9049	3680
	30	2,898	731 8	9139	3950

TABLE X (*Con.*) — LENGTHS OF ARCS OF THE PARALLEL AND THE MERIDIAN AND LOGS OF N AND R_m

[Metric Units.]

Latitude.		Parallel. Value of 1°.	Meridian. Value of 1°.	Log N	Log R_m
°	′	Meters.	Meters.		
23	00	102,524	110,738 8	6 8049231	6 8024225
	30	2,143	746 0	9323	4504
24	00	1,754	753 2	9418	4788
	30	1,357	760 6	9514	5077
25	00	100,952	110,768 0	6 8049612	6 8025370
	30	0,539	775 6	9711	5667
26	00	0,119	783.3	9812	5968
	30	99,692	791 1	9914	6274
27	00	9,257	799 0	6 8050017	6584
	30	8,814	807 0	0121	6897
28	00	8,364	815 1	0227	7215
	30	7,906	823 3	0334	7536
29	00	7,441	831 6	0443	7862
	30	6,968	840 0	0552	8190
30	00	96,488	110,848 5	6 8050663	6 8028522
	30	6,001	857 0	0774	8857
31	00	95,506	865 7	0888	9197
	30	5,004	874 4	1002	9539
32	00	4,495	883 2	1117	9883
	30	3,979	892 1	1233	6 8030231
33	00	3,455	901 1	1350	0582
	30	2,925	910 1	1468	0935
34	00	2,387	919 2	1586	1292
	30	1,842	928 3	1706	1651
35	00	91,290	110,937 6	6 8051826	6 8032012
	30	0,731	946 9	1947	2375
36	00	0,166	956 2	2069	2741
	30	89.593	965 6	2192	3109
37	00	9,014	975 1	2315	3479
	30	8,428	984 5	2439	3850
38	00	7,835	994 1	2564	4224
	30	7,235	111,003 7	2689	4599
39	00	6,629	013 3	2814	4976
	30	6,016	023 0	2940	5354
40	00	85,396	111,032 7	6 8053067	6 8035734
	30	4,770	042 4	3194	6115
41	00	4,137	052 2	3321	6496
	30	3,498	061 9	3448	6878
42	00	2,853	071 7	3576	7262
	30	2,201	081 6	3704	7646
43	00	1,543	091 4	3832	8031
	30	0,879	101 3	3960	8416
44	00	80,208	111 1	4089	8802
	30	79,532	121 0	4218	9188
45	00	78,849	111,130 9	6 8054347	6.8039574
	30	8,160	140.8	4476	9960

TABLE X (*Con.*).—LENGTHS OF ARCS OF THE PARALLEL AND THE MERIDIAN AND LOGS OF N AND R_m

[Metric Units.]

Latitude.		Parallel. Value of 1°.	Meridian. Value of 1°.	Log N.	Log R_m.
°	′	Meters.	Meters.		
46	00	77,466	111,150 6	6.8054604	6.8040346
	30	6,765	160 5	4732	0731
47	00	6,058	170 4	4861	1117
	30	5,346	180 2	4989	1502
48	00	4,628	190 1	5118	1887
	30	3,904	199 9	5246	2270
49	00	3,174	209 7	5373	2653
	30	2,439	219 5	5500	3034
50	00	71,698	111,229 3	6 8055628	6 8043416
	30	0,952	239 0	5754	3796
51	00	0,200	248 7	5880	4175
	30	69,443	258 3	6006	4552
52	00	8,680	268 0	6131	4928
	30	7,913	277 6	6256	5302
53	00	7,140	287 1	6380	5674
	30	6,361	296 6	6504	6044
54	00	5,578	306 0	6627	6413
	30	4,790	315 4	6749	6779
55	00	63,996	111,324 8	6 8056870	6 8047144
	30	3,198	334 0	6991	7506
56	00	2,395	343 3	7111	7866
	30	1,587	352 4	7230	8223
57	00	0,774	361 5	7348	8578
	30	59,957	370 5	7465	8929
58	00	9,135	379 5	7582	9279
	30	8,309	388 4	7697	9624
59	00	7,478	397 2	7811	9968
	30	6,642	405 9	7924	6 8050307
60	00	55,802	111,414 5	6 8058037	6 8050644
	30	4,958	423 1	8148	0977
61	00	4,110	431 5	8258	1307
	30	3,257	439 9	8366	1633
62	00	2,400	448 2	8474	1956
	30	1,540	456 4	8580	2274
63	00	0,675	464 4	8685	2590
	30	49,806	472 4	8789	2900
64	00	8,934	480 3	8891	3208
	30	8,057	488 1	8992	3510
65	00	47,177	111,495 7	6 8059092	6 8053809
	30	6,294	503 3	9190	4103
66	00	5,407	510 7	9287	4393
	30	4,516	518 0	9382	4678
67	00	43,622	525 3	9475	4959
	30	2,724	532 3	9567	5235
68	00	1,823	539 3	9658	5506
	30	0,919	546 2	9747	5772

TABLE X (*Con.*).—LENGTHS OF ARCS OF THE PARALLEL AND THE MERIDIAN AND LOGS OF N AND R_m

[Metric Units.]

Latitude		Parallel. Value of 1°	Meridian. Value of 1°.	Log N.	Log R_m
°	′	Meters.	Meters.		
69	00	40,012	111,552.9	6.8059834	6 8056034
	30	39,102	559 5	9919	6290
70	00	38,188	111,565 9	6 8060003	6.8056542
	30	7,272	572 2	0085	6788
71	00	6,353	578 4	0165	7029
	30	5,421	584 5	0244	7264
72	00	4,506	590 4	0321	7495
	30	3,578	596 2	0396	7719
73	00	2,648	601 8	0468	7938
	30	1,716	607 3	0539	8153
74	00	0,781	612 7	0608	8361
	30	29,843	617 9	0676	8563
75	00	28,903	111,622 9	6 8060742	6 8058759
	30	7,961	627 8	0805	8950
76	00	7,017	632 6	0867	9135
	30	6,071	637 1	0927	9314
77	00	5,123	641 6	0984	9487
	30	4,172	645 9	1040	9653
78	00	3,220	650 0	1093	9814
	30	2,266	653 9	1145	9968
79	00	1,311	657 8	1195	6 8060118
	30	20,353	661 4	1242	0258
80	00	19,394	111,664 9	6 8061287	6 8060394
	30	8,434	668 2	1330	0523
81	00	7,472	671 1	1371	0646
	30	6,509	674 4	1409	0763
82	00	5,545	677 2	1446	0873
	30	4,579	679 9	1480	0976
83	00	3,612	682 4	1513	1074
	30	2,644	684 7	1544	1163
84	00	1,675	686 9	1571	1248
	30	10,706	688 9	1597	1325
85	00	9,735	111,690 7	6 8061620	6.8061395
	30	8,764	692 3	1642	1459
86	00	7,792	693.8	1661	1517
	30	6,819	695 1	1678	1567
87	00	5,846	696 2	1692	1611
	30	4,872	697 2	1705	1648
88	00	3,898	697 9	1715	1679
	30	2,924	698 6	1723	1702
89	00	1,949	699 0	1728	1719
	30	975	699 3	1731	1729
90	00	0	111,699 3	6.8061733	6 8061733

TABLE XI. — TABLE OF LOGARITHMS OF RADII OF CURVATURE OF THE EARTH'S SURFACE IN METERS FOR VARIOUS LATITUDES AND AZIMUTHS

[Based upon Clarke's Ellipsoid of Rotation (1866).]

Azimuth	0° lat.	1° lat.	2° lat.	3° lat.	4° lat.	5° lat.	6° lat.
° Meridian.	6.80175	6.80175	6.80175	6.80176	6.80177	6.80178	6.80180
5	177	177	178	178	179	180	182
10	184	184	184	185	186	187	188
15	195	195	195	196	197	198	199
20	209	209	210	210	211	212	214
25	227	228	228	228	229	230	232
30	248	249	249	250	250	251	252
35	272	272	272	273	273	274	276
40	296	297	297	297	298	299	300
45	322	322	322	323	324	324	325
50	348	348	348	348	349	350	351
55	373	373	373	373	374	374	375
60	396	396	396	396	397	398	398
65	417	417	417	418	418	418	419
70	435	435	436	436	436	437	437
75	450	450	450	450	451	451	452
80	461	461	461	461	462	462	463
85	468	468	468	468	468	469	469
90	470	470	470	470	471	471	472

Azimuth	6° lat.	7° lat.	8° lat.	9° lat.	10° lat.	11° lat.	12° lat.
° Meridian.	6.80180	6.80181	6.80183	6.80186	6.80188	6.80191	6.80194
5	182	184	186	188	190	193	196
10	188	190	192	194	197	200	202
15	199	201	203	205	207	210	213
20	214	215	217	219	222	224	227
25	232	233	235	237	239	242	244
30	252	254	256	257	260	262	264
35	276	277	278	280	282	284	287
40	300	301	303	304	306	308	310
45	325	326	328	329	331	333	335
50	351	352	353	354	356	358	359
55	375	376	377	379	380	382	383
60	398	399	400	401	403	404	406
65	419	420	421	422	423	424	426
70	437	438	439	440	441	442	443
75	452	452	453	454	455	456	457
80	463	463	464	465	466	467	468
85	469	470	470	471	472	473	474
90	472	472	473	474	474	475	476

TABLE XI (*Con.*).—TABLE OF LOGARITHMS OF RADII OF CURVATURE OF THE EARTH'S SURFACE IN METERS FOR VARIOUS LATITUDES AND AZIMUTHS

[Based upon Clarke's Ellipsoid of Rotation (1866).]

Azimuth.	12° lat.	13° lat.	14° lat.	15° lat.	16° lat.	17° lat.	18° lat.
°							
Meridian.	6 80194	6 80197	6 80201	6 80204	6.80208	6 80213	6 80217
5	196	199	203	206	210	215	219
10	202	206	209	213	217	221	225
15	213	216	219	223	227	231	235
20	227	230	233	236	240	244	248
25	244	247	250	254	257	261	265
30	264	267	270	273	276	280	284
35	287	289	292	295	298	301	305
40	310	313	315	318	321	324	327
45	335	337	339	342	344	347	350
50	359	361	364	366	368	371	373
55	383	385	387	389	391	394	396
60	406	407	409	411	413	415	417
65	426	427	429	430	432	434	436
70	443	444	446	447	449	451	453
75	457	458	460	461	463	464	466
80	468	469	470	471	473	474	476
85	474	475	476	478	479	480	482
90	476	477	478	480	481	482	484

Azimuth.	18° lat.	19° lat.	20° lat.	21° lat.	22° lat	23° lat.	24° lat
°							
Meridian.	6 80217	6 80222	6 80226	6 80232	6 80237	6 80242	6 80248
5	219	224	228	234	239	244	250
10	225	230	234	239	244	250	255
15	235	239	244	249	254	259	264
20	248	252	257	262	266	271	277
25	265	269	273	277	282	287	292
30	284	287	292	296	300	305	309
35	305	308	312	316	320	324	329
40	327	330	334	338	341	345	350
45	350	353	357	360	364	367	371
50	373	376	379	382	386	389	392
55	396	398	401	404	407	410	413
60	417	419	422	424	427	430	432
65	436	438	440	443	445	448	450
70	453	454	456	459	461	463	465
75	466	468	470	472	473	476	478
80	476	478	479	481	483	485	487
85	482	483	485	487	489	490	492
90	484	485	487	489	490	492	494

TABLE XI (*Con.*). — TABLE OF LOGARITHMS OF RADII OF CURVATURE OF THE EARTH'S SURFACE IN METERS FOR VARIOUS LATITUDES AND AZIMUTHS

[Based upon Clarke's Ellipsoid of Rotation (1866).]

Azimuth.	24° lat	25° lat	26° lat	27° lat.	28° lat	29° lat	30° lat.
°							
Meridian	6 80248	6 80254	6 80260	6 80266	6 80272	6 80279	6 80285
5	250	256	262	268	274	280	287
10	255	261	267	273	279	285	292
15	264	270	276	282	288	294	300
20	277	282	288	293	299	305	311
25	292	297	302	308	313	319	325
30	309	314	319	324	330	335	340
35	329	333	338	343	348	353	358
40	350	354	358	362	367	372	377
45	371	375	379	383	387	391	396
50	392	396	399	403	407	411	415
55	413	416	420	423	426	430	434
60	432	435	438	442	445	448	451
65	450	453	455	458	461	464	467
70	465	468	470	473	475	478	481
75	478	480	482	484	487	489	492
80	487	489	491	493	495	498	500
85	492	494	496	498	501	503	505
90	494	496	498	500	502	504	507

Azimuth	30° lat	31° lat	32° lat	33° lat	34° lat	35° lat	36° lat
°							
Meridian	6 80285	6 80292	6 80299	6 80306	6 80313	6 80320	6 80327
5	287	294	300	307	314	322	329
10	292	298	305	312	319	326	333
15	300	306	313	320	326	333	340
20	311	317	324	330	337	343	350
25	325	331	337	343	349	355	362
30	340	346	352	358	364	370	376
35	358	363	369	374	380	385	391
40	377	382	386	392	397	402	407
45	396	400	405	410	414	419	424
50	415	419	423	428	432	436	441
55	434	437	441	445	449	453	457
60	451	455	458	462	465	469	472
65	467	470	473	476	480	483	486
70	481	484	486	489	492	495	498
75	492	494	497	500	502	505	508
80	500	502	505	507	510	512	515
85	505	507	510	512	514	517	519
90	507	509	511	514	516	518	521

TABLE XI (*Con.*). — TABLE OF LOGARITHMS OF RADII OF CURVATURE OF THE EARTH'S SURFACE IN METERS FOR VARIOUS LATITUDES AND AZIMUTHS

[Based upon Clarke's Ellipsoid of Rotation (1866).]

Azimuth.	36° lat.	37° lat.	38° lat	39° lat.	40° lat.	41° lat.	42° lat.
° Meridian.	6 80327	6 80335	6.80342	6.80350	6.80357	6.80365	6.80373
5	329	336	344	351	359	366	374
10	333	340	348	355	363	370	378
15	340	348	355	362	369	376	384
20	350	357	364	371	378	385	392
25	362	368	375	382	388	395	402
30	376	382	388	394	401	407	413
35	391	397	402	408	414	420	426
40	407	412	418	423	429	434	440
45	424	429	434	439	444	449	454
50	441	445	450	454	459	464	468
55	457	461	465	469	474	478	482
60	472	476	480	484	487	491	495
65	486	489	493	496	500	503	507
70	498	501	504	507	510	514	517
75	508	510	513	516	519	522	525
80	515	517	520	523	525	528	531
85	519	522	524	527	529	532	534
90	521	523	526	528	531	533	536

Azimuth.	42° lat.	43° lat.	44° lat	45° lat.	46° lat.	47° lat.	48° lat.
° Meridian.	6 80373	6 80380	6 80388	6 80396	6.80404	6 80411	6 80419
5	374	382	389	397	404	412	420
10	378	385	393	400	408	415	423
15	384	391	398	406	413	420	428
20	392	399	406	413	420	427	434
25	402	408	415	422	429	436	442
30	413	420	426	433	439	446	452
35	426	432	438	444	450	456	462
40	440	446	451	457	462	468	474
45	454	459	464	470	475	480	485
50	468	473	478	482	487	492	496
55	482	486	490	495	499	503	508
60	495	499	502	506	510	514	518
65	507	510	514	517	520	524	528
70	517	520	523	526	529	532	536
75	525	528	531	534	536	539	542
80	531	534	536	539	542	544	547
85	534	537	540	542	545	548	550
90	536	538	541	544	546	549	551

TABLE XI (*Con.*). — TABLE OF LOGARITHMS OF RADII OF CURVATURE OF THE EARTH'S SURFACE IN METERS FOR VARIOUS LATITUDES AND AZIMUTHS

[Based upon Clarke's Ellipsoid of Rotation (1866).]

Azimuth.	48° lat.	49° lat	50° lat	51° lat	52° lat	53° lat	54° lat
° Meridian.	6 80419	6 80426	6 80434	6 80442	6 80449	6 80457	6 80464
5	420	428	435	443	450	458	465
10	423	430	438	445	453	460	467
15	428	435	442	450	457	464	471
20	434	441	448	455	462	469	476
25	442	449	456	463	469	476	482
30	452	458	465	471	477	484	490
35	462	468	474	480	486	492	498
40	474	479	485	490	496	501	506
45	485	490	495	500	505	510	515
50	496	501	506	510	515	520	524
55	508	512	516	520	524	528	533
60	518	522	526	530	533	537	541
65	528	531	534	538	541	545	548
70	536	539	542	545	548	551	554
75	542	545	548	551	554	557	559
80	547	550	553	555	558	561	563
85	550	553	555	558	560	563	566
90	551	554	556	559	561	564	566

Azimuth	54° lat.	55° lat	56° lat	57° lat	58° lat	59° lat	60° lat
° Meridian.	6 80464	6 80471	6.80479	6 80486	6 80493	6 80500	6 80506
5	465	472	479	486	493	500	07
10	467	474	481	488	495	502	09
15	471	478	485	492	498	505	11
20	476	483	489	496	502	509	15
25	482	489	495	501	508	514	20
30	490	496	502	508	514	519	25
35	498	503	509	515	520	525	31
40	506	512	517	522	527	532	37
45	515	520	525	530	534	539	43
50	524	528	533	537	542	546	50
55	533	537	541	545	548	552	56
60	541	544	548	552	555	558	62
65	548	551	555	558	561	564	67
70	554	557	560	563	566	569	72
75	559	562	565	568	570	573	75
80	563	566	568	571	573	576	78
85	566	568	570	573	575	578	80
90	566	569	571	574	576	578	80

TABLE XI (*Con.*). — TABLE OF LOGARITHMS OF RADII OF CURVATURE OF THE EARTH'S SURFACE IN METERS FOR VARIOUS LATITUDES AND AZIMUTHS

[Based upon Clarke's Ellipsoid of Rotation (1866).]

Azimuth.	60° lat.	61° lat.	62° lat.	63° lat	64° lat	65° lat	66° lat.
°							
Meridian.	6 80506	6 80513	6 80520	6 80526	6 80532	6 80538	6 80544
5	07	14	20	26	32	38	44
10	09	15	22	28	34	40	45
15	11	18	24	30	36	42	47
20	15	21	27	33	39	44	50
25	20	26	31	37	42	48	53
30	25	30	36	41	46	51	56
35	31	36	41	46	51	56	60
40	37	42	46	51	56	60	64
45	43	48	52	56	60	64	68
50	50	54	58	62	65	69	73
55	56	60	63	67	70	74	77
60	62	65	68	72	75	78	81
65	67	70	73	76	79	82	84
70	72	74	77	80	82	85	87
75	75	78	80	83	85	87	90
80	78	80	83	85	87	89	91
85	80	82	84	86	88	90	92
90	80	83	85	87	89	91	93

Azimuth	66° lat.	67° lat	68° lat	69° lat	70° lat.	71° lat	72° lat
°							
Meridian.	6 80544	6 80550	6 80555	6 80560	6 80565	6 80570	6 80575
5	44	50	55	61	66	70	75
10	45	51	56	62	66	71	76
15	47	53	58	63	68	72	77
20	50	55	60	65	70	74	78
25	53	58	62	67	72	76	80
30	56	61	65	70	74	78	82
35	60	64	69	73	77	81	84
40	64	68	72	76	80	83	87
45	68	72	76	79	83	86	89
50	73	76	79	83	86	89	92
55	77	80	83	86	89	91	94
60	81	84	86	89	91	94	96
65	84	87	89	92	94	96	98
70	87	90	92	94	96	98	6 80600
75	90	92	94	96	98	6 80600	01
80	91	93	95	97	99	01	02
85	92	94	96	98	6 80600	01	03
90	93	95	97	98	00	02	03

TABLE XII.—VALUES OF LOG m FOR COMPUTING SPHERICAL EXCESS. (METRIC SYSTEM.)

Latitude ° ′	Log m	Latitude ° ′	Log m	Latitude ° ′	Log m
18 00	1.40639 − 10	33 00	1.40520 − 10	48 00	1.40369 − 10
18 30	636	33 30	516	48 30	364
19 00	632	34 00	511	49 00	359
19 30	629	34 30	506	49 30	354
20 00	626	35 00	501	50 00	349
20 30	623	35 30	496	50 30	344
21 00	619	36 00	491	51 00	339
21 30	616	36 30	486	51 30	334
22 00	612	37 00	482	52 00	329
22 30	608	37 30	477	52 30	324
23 00	605	38 00	472	53 00	319
23 30	601	38 30	467	53 30	314
24 00	597	39 00	462	54 00	309
24 30	594	39 30	457	54 30	304
25 00	590	40 00	452	55 00	299
25 30	586	40 30	446	55 30	295
26 00	582	41 00	441	56 00	290
26 30	578	41 30	436	56 30	285
27 00	573	42 00	431	57 00	280
27 30	569	42 30	426	57 30	276
28 00	565	43 00	421	58 00	271
28 30	560	43 30	416	58 30	266
29 00	556	44 00	411	59 00	262
29 30	552	44 30	406	59 30	257
30 00	548	45 00	400	60 00	253
30 30	544	45 30	395	60 30	249
31 00	539	46 00	390	61 00	244
31 30	534	46 30	385	61 30	240
32 00	530	47 00	380	62 00	235
32 30	1 40525	47 30	1 40375	62 30	1.40231

(The above table is computed for the Clarke spheroid of 1866.)

TABLE XIVa.—LOG F (—20)

Lat.	Log F	Lat	Log F	Lat	Log F	Lat	Log F	Lat	Log F	Lat	Log F
18°	7.738	24°	7.823	30°	7.866	36°	7.877	42°	7.860	48°	7.814
19	7.756	25	7.832	31	7.870	37	7.876	43	7.854	49	7.804
20	7.772	26	7.841	32	7.873	38	7.874	44	7.848	50	7.792
21	7.787	27	7.849	33	7.875	39	7.872	45	7.840	51	7.780
22	7.800	28	7.855	34	7.877	40	7.869	46	7.832	52	7.767
23	7.812	29	7.861	35	7.877	41	7.864	47	7.824	53	7.753

TABLE XIII.—CORRECTION TO LONGITUDE FOR DIFFERENCE BETWEEN ARC AND SINE

log s (−).	log difference.	log $d\lambda$ (+)	log s (−).	log difference	log $d\lambda$ (+).
3 876	0 000 0001	2 385	4 871	0.000 0098	3.380
4 026	02	2 535	4.882	103	3.391
4 114	03	2 623	4.892	108	3 401
4 177	04	2 686	4 903	114	3.412
4 225	05	2 734	4.913	119	3.422
4 265	06	2.774	4 922	124	3.431
4 298	07	2 807	4 932	130	3 441
4 327	08	2 836	4 941	136	3 450
4 353	09	2.862	4 950	142	3.459
4.376	10	2 885	4.959	147	3.468
4 396	11	2 905	4 968	153	3 477
4.415	12	2 924	4 976	160	3 485
4.433	13	2 942	4 985	166	3 494
4 449	14	2.958	4 993	172	3 502
4.464	15	2 973	5 002	179	3 511
4.478	16	2.987	5 010	186	3.519
4 491	17	3 000	5 017	192	3.526
4 503	18	3 012	5 025	199	3 534
4 526	20	3 035	5 033	206	3 542
4.548	23	3.057	5.040	213	3.549
4.570	25	3 079	5 047	221	3 556
4.591	27	3.100	5 054	228	3 563
4.612	30	3 121	5 062	236	3.571
4.631	33	3 140	5 068	243	3 577
4 649	36	3 158	5 075	251	3 584
4 667	39	3 176	5.082	259	3 591
4 684	42	3 193	5 088	267	3 597
4 701	45	3 210	5 095	275	3 604
4 716	48	3 225	5 102	284	3 611
4 732	52	3.241	5 108	292	3 617
4 746	56	3.255	5 114	300	3.623
4 761	59	3 270	5 120	309	3 629
4 774	63	3.283	5.126	318	3 635
4.788	67	3 297	5 132	327	3 641
4 801	71	3.310	5 138	336	3 647
4.813	75	3.322	5.144	345	3.653
4.825	80	3 334	5 150	354	3.659
4.834	84	3.343	5.156	364	3.665
4.849	89	3.358	5.161	373	3.670
4.860	94	3.369	5.167	383	3 676

TABLE XIV.—LOGARITHMS OF FACTORS FOR COMPUTING GEODETIC POSITIONS

Lat.	Log *A*	Log *B*	Log *C*	Log *D*	Log *E*
° ′	−10	−10	−10	−10	−20
18 00	8.509 5862	8.512 2550	0.91816	2.1606	5.7317
10	5836	2474	0.92243	2.1641	5.7337
20	5811	2397	0.92667	2.1675	5.7358
30	5785	2320	0.93088	2.1709	5.7379
40	5759	2243	0.93505	2.1742	5.7400
50	5733	2165	0.93919	2.1775	5.7422
19 00	5707	2086	0.94330	2.1808	5.7443
10	5681	2006	0.94737	2.1840	5.7464
20	5654	1927	0.95142	2.1872	5.7486
30	5627	1847	0.95544	2.1903	5.7508
40	5600	1766	0.95943	2.1934	5.7530
50	5573	1684	0.96339	2.1965	5.7552
20 00	5546	1602	0.96733	2.1996	5.7574
10	5518	1519	0.97123	2.2026	5.7597
20	5490	1435	0.97511	2.2055	5.7619
30	5462	1351	0.97896	2.2084	5.7642
40	5434	1267	0.98279	2.2113	5.7664
50	5406	1182	0.98659	2.2142	5.7688
21 00	5377	1096	0.99037	2.2170	5.7711
10	5348	1010	0.99412	2.2198	5.7734
20	5320	0924	0.99785	2.2226	5.7757
30	5290	0836	1.00156	2.2253	5.7780
40	5261	0748	1.00524	2.2280	5.7804
50	5232	0660	1.00890	2.2307	5.7828
22 00	5202	0571	1.01253	2.2333	5.7851
10	5172	0481	1.01615	2.2359	5.7875
20	5142	0391	1.01974	2.2385	5.7899
30	5112	0301	1.02331	2.2411	5.7924
40	5082	0210	1.02686	2.2436	5.7948
50	5051	0118	1.03039	2.2461	5.7972
23 00	5020	8.512 0026	1.03390	2.2485	5.7997
10	4990	8.511 9934	1.03739	2.2510	5.8021
20	4959	9840	1.04086	2.2534	5.8046
30	4927	9747	1.04431	2.2557	5.8071
40	4896	9653	1.04775	2.2581	5.8096
50	4865	9558	1.05116	2.2604	5.8121
24 00	4833	9463	1.05456	2.2627	5.8146
10	4801	9367	1.05794	2.2650	5.8172
20	4769	9271	1.06130	2.2672	5.8197
30	4737	9174	1.06464	2.2694	5.8223
40	4704	9077	1.06797	2.2716	5.8249
50	4672	8979	1.07128	2.2738	5.8274
60	8.509 4639	8.511 8881	1.07457	2.2759	5.8300

TABLE XIV (*Continued*)

Lat.	Log *A*	Log *B*	Log *C*	Log *D*	Log *E*
° ′					
25 00	8.509 4639	8.511 8881	1.07457	2 2759	5.8300
10	4606	8783	1.07785	2.2780	5.8326
20	4573	8684	1.08111	2.2801	5.8352
30	4540	8584	1.08435	2.2822	5.8379
40	4507	8484	1.08758	2.2842	5.8405
50	4473	8383	1.09080	2.2862	5.8431
26 00	4439	8283	1.09400	2.2882	5.8458
10	4406	8181	1 09718	2.2902	5.8485
20	4372	8079	1.10036	2.2922	5.8512
30	4337	7977	1.10351	2.2941	5 8539
40	4303	7874	1.10666	2.2960	5 8566
50	4269	7771	1.10979	2.2978	5.8593
27 00	4234	7667	1.11290	2.2997	5.8620
10	4200	7563	1.11600	2.3015	5 8647
20	4165	7458	1 11909	2 3033	5 8675
30	4130	7353	1.12217	2 3051	5 8702
40	4094	7248	1.12523	2 3069	5.8730
50	4059	7142	1.12829	2.3086	5 8757
28 00	4024	7036	1.13132	2.3104	5.8785
10	3988	6929	1 13435	2.3121	5.8813
20	3952	6822	1 13737	2.3137	5.8841
30	3917	6714	1 14037	2.3154	5.8870
40	3881	6607	1 14337	2.3170	5.8898
50	3845	6498	1 14635	2.3187	5.8926
29 00	3808	6389	1 14932	2 3203	5.8955
10	3772	6280	1.15228	2.3218	5.8983
20	3735	6171	1.15522	2 3234	5.9012
30	3699	6061	1.15816	2 3249	5.9041
40	3662	5950	1.16109	2.3264	5.9069
50	3625	5840	1.16401	2.3279	5.9098
30 00	3588	5729	1.16692	2 3294	5.9127
10	3551	5617	1.16981	2.3309	5.9157
20	3514	5505	1.17270	2 3323	5 9186
30	3476	5393	1.17558	2.3337	5.9215
40	3439	5281	1.17845	2.3351	5.9245
50	3401	5168	1.18131	2.3365	5.9274
31 00	3363	5054	1.18416	2.3379	5.9304
10	3325	4941	1.18700	2.3392	5 9334
20	3287	4827	1.18983	2.3405	5 9363
30	3249	4713	1.19266	2.3418	5.9393
40	3211	4598	1.19548	2.3431	5.9423
50	3173	4483	1.19828	2.3444	5.9453
60	8.509 3134	8.511 4368	1.20108	2.3456	5.9484

TABLE XIV (*Continued*)

Lat.	Log *A*	Log *B*	Log *C*	Log *D*	Log *E*
° ′					
32 00	8.509 3134	8.511 4368	1.20108	2.3456	5.9484
10	3096	4252	1.20387	2.3469	5.9514
20	3057	4136	1.20666	2.3481	5.9544
30	3018	4020	1.20944	2.3493	5.9575
40	2980	3903	1.21220	2.3504	5.9605
50	2940	3786	1.21496	2.3516	5.9636
33 00	2901	3669	1.21772	2.3527	5.9667
10	2862	3551	1.22047	2.3539	5.9698
20	2823	3433	1.22321	2.3550	5.9729
30	2784	3315	1.22594	2.3561	5.9760
40	2744	3197	1.22866	2.3571	5.9791
50	2704	3078	1.23138	2.3582	5.9822
34 00	2665	2959	1.23409	2.3592	5.9853
10	2625	2840	1.23680	2.3602	5.9885
20	2585	2720	1.23950	2.3612	5.9916
30	2545	2600	1.24219	2.3622	5.9948
40	2505	2480	1.24488	2.3632	5.9980
50	2465	2360	1.24756	2.3642	6.0011
35 00	2425	2239	1.25024	2.3651	6.0043
10	2384	2118	1.25291	2.3660	6.0075
20	2344	1997	1.25557	2.3669	6.0107
30	2304	1875	1.25823	2.3678	6.0140
40	2263	1754	1.26088	2.3687	6.0172
50	2222	1632	1.26353	2.3695	6.0204
36 00	2182	1510	1.26617	2.3704	6.0237
10	2141	1387	1.26881	2.3712	6.0269
20	2100	1265	1.27145	2.3720	6.0302
30	2059	1142	1.27407	2.3728	6.0334
40	2018	1019	1.27670	2.3735	6.0367
50	1977	0895	1.27932	2.3743	6.0400
37 00	1936	0772	1.28193	2.3750	6.0433
10	1895	0648	1.28454	2.3758	6.0466
20	1853	0524	1.28715	2.3765	6.0499
30	1812	0400	1.28975	2.3772	6.0533
40	1771	0276	1.29234	2.3779	6.0566
50	1729	0151	1.29494	2.3785	6.0600
38 00	1687	8.511 0027	1.29753	2.3792	6.0633
10	1646	8.510 9902	1.30011	2.3798	6.0667
20	1604	9777	1.30269	2.3804	6.0701
30	1562	9652	1.30527	2.3810	6.0734
40	1521	9526	1.30785	2.3816	6.0768
50	1479	9401	1.31042	2.3822	6.0802
60	8.509 1437	8.510 9275	1.31299	2.3827	6.0836

TABLE XIV (*Continued*)

Lat	Log *A*	Log *B*	Log *C*	Log *D*	Log *E*
° ′					
39 00	8 509 1437	8 510 9275	1 31299	2 .3827	6 0836
10	1395	9149	1 31555	2 3832	6 .0871
20	1353	9023	1 31811	2 3838	6 .0905
30	1311	8897	1 .32067	2 3843	6 .0939
40	1269	8771	1 32323	2 3848	6 .0974
50	1227	8644	1 .32578	2 .3852	6 .1008
40 00	1184	8517	1 .32833	2 .3857	6 .1043
10	1142	8391	1 .33088	2 .3861	6 .1078
20	1100	8264	1 33342	2 .3866	6 .1113
30	1057	8137	1 .33596	2 3870	6 .1148
40	1015	8010	1 .33850	2 .3874	6 .1183
50	0973	7883	1 .34104	2 3878	6 .1218
41 00	0930	7755	1 34358	2 .3882	6 .1253
10	0888	7628	1 34611	2 .3885	6 .1289
20	0845	7500	1 34864	2 .3889	6 .1324
30	0803	7373	1 .35117	2 .3892	6 1360
40	0760	7245	1 35370	2 .3895	6 .1395
50	0718	7117	1 .35623	2 3898	6 .1431
42 00	0675	6989	1 .35875	2 .3901	6 .1467
10	0632	6861	1 36127	2 .3903	6 .1503
20	0590	6733	1 36379	2 .3906	6 .1539
30	0547	6605	1 36631	2 .3908	6 .1575
40	0504	6477	1 .36883	2 .3910	6 1612
50	0461	6348	1 37135	2 3913	6 .1648
43 00	0419	6220	1 .37386	2 3914	6 .1684
10	0376	6092	1 37638	2 3916	6 .1721
20	0333	5963	1 37889	2 3918	6 .1758
30	0290	5835	1 38141	2 .3919	6 .1795
40	0247	5706	1 .38392	2 .3921	6 .1831
50	0204	5578	1 .38643	2 .3922	6 .1868
44 00	0162	5449	1 .38894	2 .3923	6 .1905
10	0119	5320	1 39145	2 .3924	6 .1943
20	0076	5192	1 39396	2 .3925	6 .1980
30	8 5090033	5063	1 39648	2 .3925	6 .2017
40	8 5089990	4935	1 .39898	2 .3926	6 .2055
50	9947	4806	1 .40149	2 .3926	6 .2092
45 00	9904	4677	1 .40400	2 .3926	6 .2130
10	9861	4548	1 .40651	2 .3926	6 .2168
20	9818	4420	1 .40902	2 .3926	6 .2206
30	9776	4291	1 41153	2 .3926	6 .2244
40	9733	4162	1 .41404	2 .3925	6 .2283
50	9689	4034	1 .41655	2 .3925	6 .2321
60	8.508 9647	8.510 3905	1 .41906	2 .3924	6 .2359

TABLE XIV (*Continued*)

Lat.	Log *A*	Log *B*	Log *C*	Log *D*	Log *E*
° ′					
46 00	8.508 9647	8.510 3905	1 41906	2 3924	6 2359
10	9604	3776	1 .42157	2 .3923	6 2398
20	9561	3648	1 42409	2 3922	6 2436
30	9518	3519	1 42660	2 3921	6 2475
40	9475	3391	1 42911	2 3920	6 .2514
50	9433	3262	1 43163	2 3918	6 2553
47 00	9390	3134	1 43414	2 3917	6 2592
10	9347	3005	1 43666	2 3915	6 2632
20	9304	2877	1 43917	2 3913	6 2671
30	9261	2749	1 .44169	2 3911	6 2710
40	9219	2621	1 44421	2 3909	6 2750
50	9176	2493	1 44673	2 3906	6 2790
48 00	9133	2364	1 44926	2 3904	6 2830
10	9091	2236	1 45178	2 3901	6 2870
20	9048	2108	1 45431	2 3898	6 2910
30	9005	1981	1 45683	2 3895	6 2950
40	8963	1853	1 45937	2 3892	6 2990
50	8920	1725	1 46190	2 3889	6 3031
49 00	8878	1598	1 .46443	2 .3886	6 3071
10	8835	1470	1 46696	2 3882	6 3112
20	8793	1343	1 46950	2 .3878	6 3153
30	8750	1216	1 47204	2 3875	6 3194
40	8708	1088	1 47459	2 3871	6 3235
50	8666	0962	1 47713	2 .3866	6 .3276
50 00	8623	0835	1 47968	2 3862	6 3318
10	8581	0708	1 48223	2 3858	6 3359
20	8539	0581	1 48478	2 3853	6 .3401
30	8497	0455	1 48734	2 3848	6 3443
40	8455	0328	1 48989	2 3843	6 3485
50	8413	0202	1 49246	2 3838	6 3527
51 00	8371	8.510 0076	1 49502	2 3833	6 3569
10	8329	8 509 9950	1 49759	2 3828	6 3612
20	8287	9825	1 50016	2 3822	6 3654
30	8245	9699	1 50273	2 3817	6 3697
40	8203	9574	1 50531	2 .3811	6 3740
50	8161	9448	1 50789	2 .3805	6 3782
52 00	8120	9323	1 51048	2 3799	6 3826
10	8078	9198	1 51307	2 .3792	6 3869
20	8036	9074	1 51566	2 3786	6 3912
30	7995	8949	1 51826	2 3779	6 3956
40	7953	8825	1 52086	2 3773	6 4000
50	7912	8701	1 52347	2 .3766	6 4043
53 00	7871	8577	1 52608	2 3759	6 4088
10	7829	8453	1 52869	2 3751	6 .4132
20	7788	8329	1 .53131	2 3744	6 4176
30	7747	8206	1 53393	2 3736	6 4221
40	7706	8083	1 .53656	2 3729	6 4265
50	7665	7960	1 .53919	2 .3721	6 4310
60	8.508 7624	8.509 7838	1 54183	2 .3713	6 4355

TABLE XV.—MERIDIONAL DISTANCE IN METERS FROM WHOLE DEGREE PARALLEL.

Lat.	Minutes from Whole Degree Parallel									
	1′	2′	3′	4′	5′	6′	7′	8′	9′	10′
25°	1846.1	3692.3	5538.4	7384.6	9230.7	11076.9	12923.0	14769.2	16615.4	18461.5
26	1846.4	3692.8	5539.2	7385.6	9232.0	11078.4	12924.8	14771.2	16617.7	18464.1
27	1846.7	3693.3	5540.0	7386.6	9233.3	11080.0	12926.7	14773.3	16620.0	18466.7
28	1846.9	3693.8	5540.8	7387.7	9234.6	11081.6	12928.5	14775.5	16622.5	18469.4
29	1847.2	3694.4	5541.6	7388.8	9236.0	11083.2	12930.5	14777.7	16624.9	18472.2
30	1847.5	3695.0	5542.4	7389.9	9237.4	11084.9	12932.4	14779.9	16627.4	18475.0
31	1847.8	3695.5	5543.3	7391.1	9238.9	11086.7	12934.4	14782.2	16630.0	18477.9
32	1848.1	3696.1	5544.2	7392.3	9240.3	11088.4	12936.5	14784.6	16632.7	18480.8
33	1848.4	3696.7	5545.1	7393.4	9241.8	11090.2	12938.6	14787.0	16635.4	18483.8
34	1848.7	3697.3	5546.0	7394.6	9243.3	11092.0	12940.7	14789.4	16638.1	18486.8
35	1849.0	3697.9	5546.9	7395.9	9244.9	11093.9	12942.8	14791.8	16640.8	18489.9
36	1849.3	3698.5	5547.8	7397.1	9246.4	11095.7	12945.0	14794.3	16643.6	18493.0
37	1849.6	3699.2	5548.8	7398.4	9248.0	11097.6	12947.2	14796.8	16646.5	18496.1
38	1849.9	3699.8	5549.7	7399.6	9249.6	11099.5	12949.4	14799.4	16649.3	18499.3
39	1850.2	3700.5	5550.7	7400.9	9251.2	11101.4	12951.7	14801.9	16652.2	18502.5
40	1850.5	3701.1	5551.7	7402.2	9252.8	11103.4	12953.9	14804.5	16655.1	18505.7
41	1850.9	3701.7	5552.6	7403.5	9254.4	11105.3	12956.2	14807.1	16658.0	18509.0
42	1851.2	3702.4	5553.6	7404.8	9256.0	11107.3	12958.5	14809.7	16661.0	18512.2
43	1851.5	3703.1	5554.6	7406.1	9257.7	11109.2	12960.8	14812.4	16663.9	18515.5
44	1851.9	3703.7	5555.6	7407.4	9259.3	11111.2	12963.1	14815.0	16666.9	18518.8
45	1852.2	3704.4	5556.6	7408.8	9261.0	11113.2	12965.4	14817.6	16669.9	18522.1
46	1852.5	3705.0	5557.6	7410.1	9262.6	11115.2	12967.7	14820.3	16672.8	18525.4
47	1852.8	3705.7	5558.5	7411.4	9264.3	11117.1	12970.0	14822.9	16675.8	18528.7
48	1853.2	3706.3	5559.5	7412.7	9265.9	11119.1	12972.3	14825.5	16678.7	18531.9
49	1853.5	3707.0	5560.5	7414.0	9267.5	11121.1	12974.6	14828.1	16681.7	18535.2
50	1853.8	3707.7	5561.5	7415.3	9269.2	11123.0	12976.9	14830.7	16684.6	18538.5

TABLE XVI.—COÖRDINATES OF CURVATURE (METERS)

Long	Latitudes							
	26°		27°		28°		29°	
	X	Y	X	Y	X	Y	X	Y
1′	1668.7	0.1	1654.3	0.1	1639.4	0.1	1624.0	0.1
2	3337.3	0.4	3308.5	0.4	3278.8	0.4	3248.0	0.5
3	5006.0	1.0	4962.8	1.0	4918.2	1.0	4872.0	1.0
4	6674.6	1.7	6617.1	1.7	6557.6	1.8	6496.1	1.8
5	8343.3	2.7	8271.4	2.7	8197.0	2.8	8120.1	2.9
6	10011.9	3.8	9925.7	3.9	9836.4	4.0	9744.1	4.1
7	11680.6	5.2	11579.9	5.4	11475.7	5.5	11368.1	5.6
8	13349.2	6.8	13234.2	7.0	13115.1	7.2	12992.1	7.3
9	15017.9	8.6	14888.5	8.8	14754.5	9.1	14616.1	9.3
10	16686.6	10.6	16542.8	10.9	16393.9	11.2	16240.1	11.5

Long	30°		31°		32°		33°	
1′	1608.1	0.1	1591.8	0.1	1574.9	0.1	1557.6	0.1
2	3216.3	0.5	3183.5	0.5	3149.8	0.5	3115.2	0.5
3	4824.4	1.1	4775.3	1.1	4724.8	1.1	4672.8	1.1
4	6432.6	1.9	6367.1	1.9	6299.7	1.9	6230.3	2.0
5	8040.7	2.9	7958.9	3.0	7874.6	3.0	7787.9	3.1
6	9648.8	4.2	9550.6	4.3	9449.5	4.4	9345.5	4.4
7	11257.0	5.7	11142.4	5.8	11024.4	6.0	10903.1	6.0
8	12865.1	7.5	12734.2	7.6	12599.4	7.8	12460.7	7.9
9	14473.2	9.5	14325.9	9.7	14174.3	9.8	14018.3	10.0
10	16081.4	11.7	15917.7	11.9	15749.2	12.1	15575.9	12.3

Long	34°		35°		36°		37°	
1′	1539.8	0.1	1521.5	0.1	1502.8	0.1	1483.6	0.1
2	3079.6	0.5	3043.0	0.5	3005.5	0.5	2967.1	0.5
3	4619.3	1.1	4564.5	1.1	4508.3	1.2	4450.7	1.2
4	6159.1	2.0	6086.0	2.0	6011.1	2.1	5934.2	2.1
5	7698.9	3.1	7607.5	3.2	7513.8	3.2	7417.8	3.3
6	9238.7	4.5	9129.0	4.6	9016.6	4.6	8901.4	4.7
7	10778.5	6.1	10650.5	6.2	10519.3	6.3	10384.9	6.4
8	12318.3	8.0	12172.0	8.1	12022.1	8.2	11868.5	8.3
9	13858.0	10.1	13693.5	10.3	13524.8	10.4	13352.1	10.5
10	15397.9	12.5	15215.0	12.7	15027.6	12.8	14835.6	13.0

TABLE XVI (*Con.*).—COÖRDINATES OF CURVATURE (METERS)

Long	38° X	38° Y	39° X	39° Y	40° X	40° Y	41° X	41° Y
1′	1463.9	0.1	1443.8	0.1	1423.3	0.1	1402.3	0.1
2	2927.8	0.5	2887.6	0.5	2846.5	0.5	2804.6	0.5
3	4391.7	1.2	4331.4	1.2	4269.8	1.2	4206.9	1.2
4	5855.6	2.1	5775.2	2.1	5693.0	2.1	5609.2	2.1
5	7319.6	3.3	7219.0	3.3	7116.3	3.3	7011.5	3.3
6	8783.5	4.7	8662.9	4.8	8539.6	4.8	8413.7	4.8
7	10247.4	6.4	10106.7	6.5	9962.8	6.5	9816.0	6.6
8	11711.3	8.4	11550.5	8.5	11386.1	8.5	11218.3	8.6
9	13175.2	10.6	12994.3	10.7	12809.3	10.8	12620.6	10.8
10	14639.1	13.1	14438.1	13.2	14232.6	13.3	14022.9	13.4

Long	42° X	42° Y	43° X	43° Y	44° X	44° Y	45° X	45° Y
1′	1380.9	0.1	1359.1	0.1	1336.8	0.1	1314.1	0.1
2	2761.8	0.5	2718.1	0.5	2673.6	0.5	2628.3	0.5
3	4142.7	1.2	4077.2	1.2	4010.4	1.2	3942.5	1.2
4	5523.5	2.2	5436.2	2.2	5347.2	2.2	5256.6	2.2
5	6904.4	3.4	6795.3	3.4	6684.0	3.4	6570.8	3.4
6	8285.3	4.8	8154.3	4.9	8020.8	4.9	7884.9	4.9
7	9666.2	6.6	9513.4	6.6	9357.7	6.6	9199.1	6.6
8	11047.1	8.6	10872.4	8.6	10694.5	8.6	10513.2	8.6
9	12428.0	10.9	12231.5	10.9	12031.3	10.9	11827.4	10.9
10	13808.8	13.4	13590.5	13.5	13368.1	13.5	13141.5	13.5

Long	46° X	46° Y	47° X	47° Y	48° X	48° Y	49° X	49° Y
1′	1291.1	0.1	1267.6	0.1	1243.8	0.1	1219.6	0.1
2	2582.2	0.5	2535.3	0.5	2487.6	0.5	2439.1	0.5
3	3873.3	1.2	3802.9	1.2	3731.4	1.2	3658.7	1.2
4	5164.4	2.2	5070.5	2.2	4975.2	2.1	4878.3	2.1
5	6455.5	3.4	6338.2	3.4	6219.0	3.3	6097.9	3.3
6	7746.6	4.9	7605.8	4.8	7462.8	4.8	7317.5	4.8
7	9037.6	6.6	8873.5	6.6	8706.6	6.6	8537.0	6.6
8	10328.7	8.6	10141.1	8.6	9950.4	8.6	9756.6	8.6
9	11619.8	10.9	11408.7	10.9	11194.2	10.9	10976.2	10.8
10	12910.9	13.5	12676.4	13.5	12437.9	13.4	12195.8	13.4

TABLE XVII. — COÖRDINATES OF CURVATURE (METERS)

Long	Latitudes 25° X	25° Y	30° X	30° Y	35° X	35° Y
5°	504 645	9 307	482 288	10 523	456 261	11 421
10	1 008 603	37 215	963 658	42 074	911 379	45 656
15	1 511 190	83 685	1 443 103	94 501	1 364 214	102 619
20	2 011 722	148 656	1 919 982	167 077	1 813 632	182 168
25	2 509 518	232 038	2 393 116	262 089	2 258 507	284 102
30	3 003 900	333 718	2 861 694	376 749	2 697 724	408 168

Long	40° X	40° Y	45° X	45° Y	50° X	50° Y
5°	426 757	11 072	393 996	12 160	358 224	11 978
10	852 171	47 852	786 492	48 594	714 847	47 859
15	1 274 904	107 525	1 175 994	109 162	1 068 277	107 482
20	1 693 628	190 805	1 561 019	193 635	1 416 934	190 581
25	2 107 023	297 430	1 940 103	301 690	1 759 262	296 785
30	2 513 790	427 063	2 311 802	432 918	2 093 731	425 619

APPENDIX

SPHERICAL EXCESS

Formula [61*a*] for spherical excess may be derived as follows: Starting with the identical equation

$$\tan\frac{e}{4} = \frac{\sin\frac{1}{4}(A+B+C-180^\circ)}{\cos\frac{1}{4}(A+B+C-180^\circ)}\cdot\frac{\cos\frac{1}{4}(A+B-C+180^\circ)}{\cos\frac{1}{4}(A+B-C+180^\circ)}$$

we obtain, by applying formulæ for $\cos x \sin y$ and $\cos x \cos y$ in terms of the half sum and half difference,

$$\tan\frac{e}{4} = \frac{\sin\frac{1}{2}(A+B) - \sin\frac{1}{2}(180^\circ - C)}{\cos\frac{1}{2}(A+B) - \cos\frac{1}{2}(180^\circ - C)}$$

$$= \frac{\sin\frac{1}{2}(A+B) - \cos\frac{1}{2}C}{\cos\frac{1}{2}(A+B) + \sin\frac{1}{2}C}.$$

Substituting from Delambre's Equations values of $\sin\frac{1}{2}(A+B)$ and $\cos\frac{1}{2}(A+B)$,

$$\tan\frac{e}{4} = \frac{\cos\frac{1}{2}(a-b) - \cos\frac{1}{2}c}{\cos\frac{1}{2}(a+b) + \cos\frac{1}{2}c}\cdot\frac{\cos\frac{1}{2}C}{\sin\frac{1}{2}C}.$$

Applying to this the formulæ for $\cos x + \cos y$ and $\cos x - \cos y$, we have

$$\tan\frac{e}{4} = \frac{\sin\frac{1}{4}(a-b+c)\sin\frac{1}{4}(-a+b+c)}{\cos\frac{1}{4}(a+b+c)\cos\frac{1}{4}(a+b-c)}\cdot\cot\frac{C}{2}.$$

Substituting $s = \dfrac{a+b+c}{2}$ and also putting for $\cot\frac{1}{2}C$ its equivalent in terms of the sides,

$$\tan\frac{e}{4} = \sqrt{(\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c))}$$

which is Lhuillier's theorem.

Replacing each tangent by two terms of its series, and squaring, we have

$$\tan^2 \frac{e}{4} =$$

$$\left(\frac{s}{2R} + \frac{s^3}{24R^3}\right)\left(\frac{s-a}{2R} + \frac{(s-a)^3}{24R^3}\right)\left(\frac{s-b}{2R} + \frac{(s-b)^3}{24R^3}\right)\left(\frac{s-c}{2R} + \frac{(s-c)^3}{24R^3}\right),$$

in which a, b, c now represent the sides in linear units.

The square of the area of a plane triangle $= A^2 = s(s-a)(s-b)(s-c)$.

Therefore

$$\tan^2 \frac{e}{4} = \frac{A^2}{16R^4} + \frac{A^2}{192R^6}((s-a)^2 + (s-b)^2 + (s-c)^2 + s^2)$$

$$= \frac{A^2}{16R^4} + \frac{A^2}{192R^6}(a^2 + b^2 + c^2).$$

Taking the square root

$$\tan \frac{c}{4} = \frac{A}{4R^2}\left(1 + \frac{a^2 + b^2 + c^2}{12R^2}\right)^{\frac{1}{2}}.$$

Expanding,

$$\tan \frac{c}{4} = \frac{A}{4R^2}\left(1 + \frac{a^2 + b^2 + c^2}{24R^2} + \cdots\right)$$

Therefore

$$\frac{c''}{4} \sin 1'' = \frac{A}{4R^2}\left(1 + \frac{a^2 + b^2 + c^2}{24R^2}\right),$$

or

$$'' = \frac{A}{R^2 \sin 1''}\left(1 + \frac{a^2 + b^2 + c^2}{24R^2}\right). \qquad [61a]$$

CLARKE'S FORMULÆ

Given ϕ, λ of station A (Fig. 156) and the distance s and the (interior) angle α to station B; to find ϕ', λ' of station B and the (interior) angle α'.

In Fig. 156 the heavy curve joining A and B is the plane (elliptic) curve cut from the surface of the spheroid by the vertical

plane at A and passing through B (plane AHB). The normal at B ends at K. The dotted curve is the curve cut by the plane BKA. The angle α' is the angle at B from BP to the right, to the dotted curve; it is less than the angle to the full line by the small angle ζ. The angle at B between the planes BKP and BKA is $\alpha' + \zeta$.

We will first find the angle at H subtended by the plane arc $s = AB$. Join B and H. Suppose P' to be any point on AB

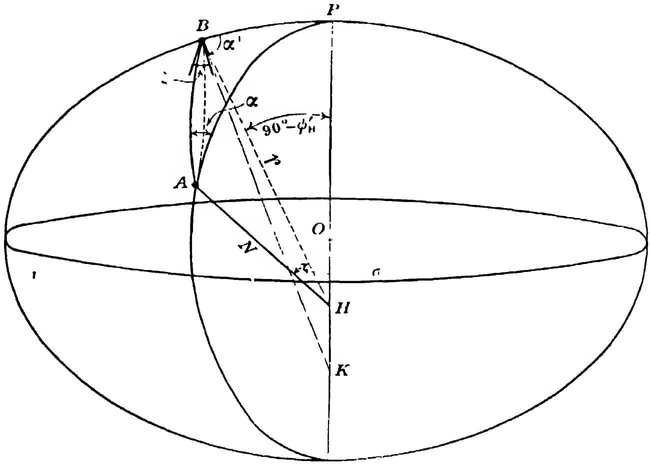


FIG. 156.

and let s represent the distance from A to P', and σ the corresponding angle at H between HA and HP'. Let $HP' = r$, $HA = N$, the *normal*, and ϕ_H' the angle between HP' and the plane of the equator; (this is nearly but not quite equal to its latitude). The coördinates of P' (in the meridian plane of P') will be

$$x = r \cos \phi_H'$$

$$y = r \sin \phi_H' - Ne^2 \sin \phi.$$

Substituting in the equation of the ellipse

$$\frac{r^2 \cos^2 \phi_H'}{a^2} + \frac{(r \sin \phi_H' - Ne^2 \sin \phi)^2}{b^2} = 1.$$

Multiplying by $a^2 = N^2(1 - e^2 \sin^2 \phi)$, then squaring the second term and multiplying, we have

$$r^2 \cos^2 \phi_H' + \frac{r^2 \sin^2 \phi_H'}{1 - e^2} - \frac{2\, e^2 rN \sin \phi_H' \sin \phi}{1 - e^2} + \frac{e^4 N^2 \sin^2 \phi}{1 - e^2}$$
$$= N^2 - N^2 e^2 \sin^2 \phi$$

in which $1 - e^2$ has been put for b^2/a^2. Substituting $1 - \sin^2 \phi_H'$ for $\cos^2 \phi_H'$, and rearranging the terms

$$r^2 + \frac{r^2 e^2 \sin^2 \phi_H'}{1 - e^2} - \frac{2\, e^2 rN \sin \phi_H' \sin \phi}{1 - e^2} + \frac{e^2 N^2 \sin^2 \phi}{1 - e^2} = N^2,$$

or, $$r^2 - N^2 = -\frac{e^2}{1 - e^2}(r \sin \phi_H' - N \sin \phi)^2.$$

Writing this in the form

$$r = \left[N^2 - \frac{e^2}{1 - e^2}(r \sin \phi_H' - N \sin \phi)^2\right]^{\frac{1}{2}}$$

and expanding,

$$r = N - \frac{1}{2\,N} \cdot \frac{e^2}{1 - e^2}(r \sin \phi_H' - N \sin \phi)^2 +$$
$$= N - \frac{e^2}{2\,(1 - e^2)}\left(\frac{r^2}{N} \sin^2 \phi_H' - 2\, r \sin \phi_H' \sin \phi + N \sin^2 \phi\right).$$

Putting $r = N$ in the small terms

$$r = N - \frac{Ne^2}{2\,(1 - e^2)}(\sin \phi_H' - \sin \phi)^2.$$

To obtain the value of ϕ_H' solve the spherical triangle formed by the three lines HA, HB, HP, the sides of which are $90° - \phi$, $90° - \phi_H'$, and σ. The angles are $\Delta\lambda$ at the pole, α at A, and $\alpha' + \zeta$ at B. Then by trigonometry,

$$\sin \phi_H' = \sin \phi \cos \sigma + \cos \phi \sin \sigma \cos \alpha.$$

Substituting in the equation for r,

$$r = N - \frac{e^2}{1 - e^2} \cdot \frac{N}{2} (\sin\phi \cos\sigma + \cos\phi \sin\sigma \cos\alpha - \sin\phi)^2.$$

Collecting the $\sin\phi$ terms and substituting for $\cos\sigma$ two terms of its series, squaring, and substituting $\sigma = \sin\sigma$, we may find,

$$\frac{r}{N} = 1 - \frac{1}{2} \frac{e^2}{1 - e^2} \left(\sigma^2 \cos^2\phi \cos^2\alpha - 2\frac{\sigma^3}{2} \sin\phi \cos\phi \cos\alpha + \cdots\right)$$
$$= 1 + P\sigma^2 + Q\sigma^3 + \cdots,$$

in which $$P = -\frac{1}{2} \cdot \frac{e^2}{1 - e^2} \cos^2\alpha \cos^2\phi,$$

and $$Q = \frac{1}{4} \cdot \frac{e^2}{1 - e^2} \cdot \cos\alpha \sin 2\phi.$$

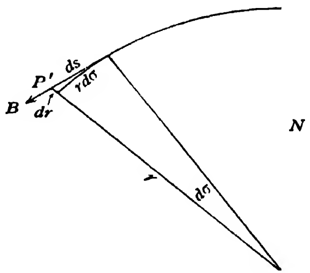


Fig 157

Referring to Fig. 157, we have from the differential triangle

$$(ds)^2 = (dr)^2 + (rd\sigma)^2$$

or $$ds = ((dr)^2 + (rd\sigma)^2)^{\frac{1}{2}}$$

and $$\frac{ds}{d\sigma} = \left(r^2 + \left(\frac{dr}{d\sigma}\right)^2\right)^{\frac{1}{2}}$$

$$= r + \frac{1}{2r}\left(\frac{dr}{d\sigma}\right)^2 +$$

Integrating,

$$s = \int_0^\sigma \left(r + \frac{1}{2r}\left(\frac{dr}{d\sigma}\right)^2 + \cdot \cdot \cdot \right) d\sigma$$

$$= \int_0^\sigma r\, d\sigma + \int_0^\sigma \frac{1}{2r}\left(\frac{dr}{d\sigma}\right)^2 d\sigma,$$

$$s = N\int_0^\sigma (1 + P\sigma^2 + \cdot \cdot \cdot)\, d\sigma + \int_0^\sigma (2NP^2\sigma^2 + \cdot \cdot \cdot)\, d\sigma,$$

$$\frac{s}{N} = \sigma + \frac{P}{3}\sigma^3 + \frac{2}{3}P^2\sigma^3 + \cdot \cdot \cdot$$

$$= \sigma + \frac{P}{3}(1 + 2P)\,\sigma^3 + \cdot \cdot \cdot \cdot .$$

Placing $\sigma = \frac{s}{N}$ in the small terms, and neglecting P^2,

$$\sigma = \frac{s}{N} + \frac{1}{6}\cdot\frac{e^2}{1 - e^2}\cdot\cos^2\phi\cos^2\alpha\cdot\frac{s^3}{N^3}.$$

Reduced to seconds this is

$$\sigma'' = \frac{s}{N \sin 1''} + \frac{e^2\sigma^3 \sin^2 1''}{6\,(1 - e^2)}\cos^2\phi\cos^2\alpha. \qquad [63a]$$

To determine ζ, the angle between the plane curves, consider the spherical triangle determined by BA, BH, BK. Figure 158 shows this triangle as it appears looking down from B. By the law of sines

$$\frac{\sin(\alpha' + \zeta)}{\sin \alpha'} = \frac{\sin ABK}{\sin ABH}.$$

In the plane triangle ABH

$$\frac{HB}{HA} = \frac{\sin HAB}{\sin HBA}$$

or

$$\frac{1}{\sin HBA} = \frac{HB}{HA \cdot \sin HAB}.$$

$$\frac{\sin(\zeta + \alpha')}{\sin \alpha'} = \frac{HB \sin ABK}{HA \sin HAB} = \frac{r \sin \mu'}{N \sin \mu}$$

in which $\mu = 90° -$ the depression angle to B and $\mu' = 90° -$ the depression angle to A. Expanding $\sin(\zeta + \alpha')$ and putting $\mu = \mu'$ (nearly), we have

$$1 + \zeta \cot \alpha' = \frac{r}{N}$$

$$= 1 - \frac{1}{2} \cdot \frac{e^2}{1 - e^2} \cdot \sigma^2 \cdot \cos^2 \alpha \cos^2 \phi.$$

Since α is nearly equal to α',

$$\zeta \cdot \frac{\cos \alpha}{\sin \alpha} = -\frac{1}{2} \cdot \frac{e^2}{1 - e^2} \cdot \sigma^2 \cdot \cos^2 \alpha \cos^2 \phi,$$

and
$$\zeta = -\frac{1}{4} \cdot \frac{e^2}{1 - e^2} \cdot \sigma^2 \cdot \sin 2\alpha \cos^2 \phi,$$

or
$$\zeta'' = -\frac{e^2 \sigma^2 \sin 1''}{4(1 - e^2)} \cdot \cos^2 \phi \sin 2\alpha.$$

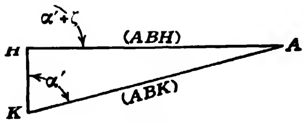


FIG. 158.

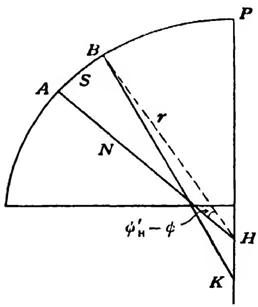


FIG. 159.

The difference in longitude is found simultaneously with the angles by the formulæ

$$\tan \tfrac{1}{2}(\alpha' + \zeta + \Delta\lambda) = \frac{\cos \frac{1}{2}(\gamma - \sigma)}{\cos \frac{1}{2}(\gamma + \sigma)} \cdot \cot \frac{\alpha}{2}$$

$$\tan \tfrac{1}{2}(\alpha' + \zeta - \Delta\lambda) = \frac{\sin \frac{1}{2}(\gamma - \sigma)}{\sin \frac{1}{2}(\gamma + \sigma)} \cdot \cot \frac{\alpha}{2}.$$

In order to find the difference in latitude we make the assumption that

$$\frac{S}{s} = \frac{\phi_H' - \phi}{\sigma} \quad \text{(nearly)}.$$

Clarke (*Geodesy*, p. 275) shows the error to be $\frac{1}{48} e^2\sigma^3 \sin^2 2\alpha \sin 2\phi$. S is the distance between the parallels of A and B.

If we revolve the plane of the meridian of B (Fig. 156) into the plane of the meridian of A we shall have Fig. 159. If we find $\phi_H' - \phi$ by the same process used in finding σ we obtain

$$\phi_H' - \phi = \frac{S}{N}\left(1 + \frac{1}{6} \cdot \frac{e^2}{1 - e^2} \cdot \cos^2 \alpha \cos^2 \phi \left(\frac{S}{N}\right)^2\right)$$

The angle $\phi_H' - \phi$ may be found from the spherical formula,

$$\tan \tfrac{1}{2} (\phi_H' - \phi) = \frac{\sin \frac{1}{2} (\alpha' + \zeta - \alpha)}{\sin \frac{1}{2} (\alpha' + \zeta + \alpha)} \tan \frac{\sigma}{2}$$

$$= k \tan \frac{\sigma}{2}.$$

whence $$\tfrac{1}{2} (\phi_H' - \phi) = \tan^{-1} \left(k \tan \frac{\sigma}{2}\right).$$

Substituting for $\frac{\sigma}{2}$ its series, and for $\tan^{-1}\left(k \tan \frac{\sigma}{2}\right)$ its series,

$$\tfrac{1}{2} (\phi_H' - \phi) = k\left(\frac{\sigma}{2} + \frac{\sigma^3}{24}\right) - \tfrac{1}{3}\left(k\frac{\sigma}{2} + k\frac{\sigma^3}{24}\right)^3 + \cdots$$

$$= k\frac{\sigma}{2}\left(1 + \frac{\sigma^2}{12}\right) - \tfrac{1}{3} k^3 \frac{\sigma^3}{8} + \cdots.$$

$$\phi_H - \phi' = k\sigma\left(1 + \frac{\sigma^2}{12}(1 - k^2)\right).$$

$$\therefore \quad S = sk\left(1 + \frac{\sigma^2}{12}(1 - k^2)\right)$$

$$= s\frac{\sin \frac{1}{2} (\alpha' + \zeta - \alpha)}{\sin \frac{1}{2} (\alpha' + \zeta + \alpha)}\left(1 + \frac{\sigma^2}{12} \cos^2 \tfrac{1}{2} (\alpha' - \alpha)\right).$$

The true difference of latitude, $\phi' - \phi$, may be found by dividing S by the radius of curvature of the meridian for the middle latitude, R_M. The error in this assumption is

$$-\left(\frac{\alpha}{2}\right)^3 \cdot e^2 \cos 2\,\phi.$$

The final formula for latitude is

$$\phi' - \phi = \frac{s}{R_M \sin 1''} \cdot \frac{\sin \frac{1}{2}(\alpha' + \zeta - \alpha)}{\sin \frac{1}{2}(\alpha' + \zeta + \alpha)} \cdot \left(1 + \frac{\sigma^2}{12} \cos^2 \tfrac{1}{2}(\alpha' - \alpha)\right).$$

INDEX

A

B

U